

SEMIDUALITIES FROM PRODUCTS OF TREES

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ABSTRACT. Let K be a global function field of characteristic p , and let Γ be a finite-index subgroup of an arithmetic group defined with respect to K and such that any torsion element of Γ is a p -torsion element. We define *semiduality groups*, and we show that Γ is a $\mathbb{Z}[1/p]$ -semiduality group if Γ acts as a lattice on a product of trees. We also give other examples of semiduality groups, including lamplighter groups, Diestel-Leader groups, and countable sums of finite groups.

1. INTRODUCTION

1.1. Arithmetic groups. Let K be a global field (number or function field), and let S be a nonempty set of finitely many inequivalent valuations of K including each archimedean valuation. The ring $\mathcal{O}_S \subseteq K$ will denote the corresponding ring of S -integers. For any $v \in S$, we let K_v be the completion of K with respect to v so that K_v is a locally compact field.

We let \mathbf{G} be a noncommutative, absolutely almost simple algebraic K -group, so that $\mathbf{G}(\mathcal{O}_S)$ is a lattice, included diagonally, in the product of simple Lie groups $\prod_{v \in S} \mathbf{G}(K_v)$. For each $v \in S$, we let X_v be the symmetric space or Euclidean building (depending on whether K_v is an archimedean or nonarchimedean field) associated with $\mathbf{G}(K_v)$, and we let $X_S = \prod_{v \in S} X_v$ so that $\mathbf{G}(\mathcal{O}_S)$ acts on X_S as a lattice.

We let

$$k(\mathbf{G}, S) = \sum_{v \in S} \text{rank}_{K_v} \mathbf{G}$$

If \mathbf{G} is K -anisotropic — that is, if $\mathbf{G}(\mathcal{O}_S)$ acts cocompactly on X_S — then there is a finite-index subgroup of $\mathbf{G}(\mathcal{O}_S)$ that is a duality group, and if K_v is an archimedean field — that is, if X_v is a symmetric space — for all $v \in S$, then there is a finite-index subgroup of $\mathbf{G}(\mathcal{O}_S)$ that is a Poincaré duality group.

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Borel-Serre [5] [6] showed that $\mathbf{G}(\mathcal{O}_S)$ is also a virtual duality group when \mathbf{G} is K -isotropic, as long as K is a number field. In particular, Borel-Serre construct an augmentation of X_S , which we denote as \widehat{X}_S , on which $\mathbf{G}(K)$ acts and $\mathbf{G}(\mathcal{O}_S)$ acts properly and cocompactly, and such that the compactly supported cohomology groups $H_c^*(\widehat{X}_S; \mathbb{Z})$ are nontrivial in some single dimension, $\ell(\mathbf{G}, S)$. The result is that any finite-index torsion-free subgroup of $\mathbf{G}(\mathcal{O}_S)$ is a duality group of dimension $\ell(\mathbf{G}, S)$ with dualizing module $H_c^{\ell(\mathbf{G}, S)}(\widehat{X}_S; \mathbb{Z})$.

The purpose of this paper is to suggest a possible analogue of Borel-Serre for arithmetic groups $\mathbf{G}(\mathcal{O}_S)$ when K is a global function field.

1.2. Function field case. Throughout the remainder of this paper, K denotes a global function field of characteristic p , and we suppose that \mathbf{G} is K -isotropic — that is, that $\mathbf{G}(\mathcal{O}_S)$ does not act cocompactly on X_S .

Any finite-index subgroup of $\mathbf{G}(\mathcal{O}_S)$ contains torsion, so it cannot be a duality group, as duality groups have finite cohomological dimension. However, there are finite-index subgroups of $\mathbf{G}(\mathcal{O}_S)$ whose only torsion elements are p -elements and whose cohomological dimension over $\mathbb{Z}[1/p]$ is bounded above by $k(\mathbf{G}, S)$. We let Γ denote such a subgroup.

The group Γ still has an obstruction to being a $\mathbb{Z}[1/p]$ -duality group. Indeed, it is not of type $FP_{k(\mathbf{G}, S)}$ over $\mathbb{Z}[1/p]$ (see Kropholler [15], Bux-Wortman [11], Gandini [13], and Bux-Köhl-Witzel [10]). However, Γ is of type $FP_{k(\mathbf{G}, S)-1}$, and we conjecture that the discrepancy between type $FP_{k(\mathbf{G}, S)}$ and $FP_{k(\mathbf{G}, S)-1}$ is the only, and in some ways a minor, obstruction to Γ being a $\mathbb{Z}[1/p]$ -duality group. Before making this precise, we'll need a definition.

For a commutative ring R , we say that a group Λ is an R -semiduality group of dimension d if

- (i) $\mathrm{cd}_R(\Lambda) \leq d$,
- (ii) Λ is of type FP_{d-1} over R ,
- (iii) $H^*(\Lambda; R\Lambda)$ is concentrated in dimension d , and
- (iv) $H^d(\Lambda; R\Lambda)$ is a flat R -module.

In the above definition, $H^d(\Lambda; R\Lambda)$ is called the *dualizing module*, and if the ring R and the group Λ are understood, then we'll often denote the dualizing module simply as D .

In Section 2 of this paper we'll show the following consequence of a group being a semiduality group.

Proposition 1. *If Λ is an R -duality group of dimension d , then for any $0 \leq n \leq d$ and any $R\Lambda$ -module M , there exist maps*

$$\varphi_n^M : H_n(\Lambda; D \otimes_R M) \rightarrow H^{d-n}(\Lambda; M)$$

such that if $\cdots \rightarrow Q_1 \rightarrow Q_0 \rightarrow M \rightarrow 0$ is a projective resolution of M by $R\Lambda$ -modules, then φ_n^M is

- (i) *surjective if Q_n and Q_{n-1} are finitely-generated,*
- (ii) *injective if Q_{n+1} and Q_n are finitely-generated,*
- (iii) *bijective if Q_{n+1} , Q_n , and Q_{n-1} are finitely-generated,*
- (iv) *injective if M is flat and $n = 1$, and*
- (v) *bijective if M is flat and $n \geq 2$.*

With the definition of semiduality and its immediate consequences listed above, we propose the following

Conjecture 2. *Let \mathcal{O}_S be a ring of S -integers in a global function field K of characteristic p , and let \mathbf{G} be a noncommutative, absolutely almost simple algebraic K -group. If Γ is a finite-index subgroup of $\mathbf{G}(\mathcal{O}_S)$ such that any torsion element of Γ is a p -element, then Γ is a $\mathbb{Z}[1/p]$ -semiduality group of dimension $k(\mathbf{G}, S)$, and the dualizing module admits an action by $\mathbf{G}(K)$.*

Furthermore, we conjecture that if D is the dualizing module $H^{k(\mathbf{G}, S)}(\Gamma; \mathbb{Z}[1/p]\Gamma)$, then D contains, and is an inverse limit of quotients of, $H_c^{k(\mathbf{G}, S)}(X_S; \mathbb{Z}[1/p])$. That is, we can view D as an augmentation of $H_c^{k(\mathbf{G}, S)}(X_S; \mathbb{Z}[1/p])$. Thus, whereas Borel-Serre exhibits duality groups whose dualizing modules are cohomology groups of augmentations of the spaces on which arithmetic groups act, we conjecture that over function fields, arithmetic groups are semiduality groups whose dualizing modules are augmentations of cohomology groups of spaces on which the arithmetic groups act.

As an illustration, let L be a field whose characteristic is not equal to p . Recall that $\text{cd}_L(\Gamma) \leq k(\mathbf{G}, S)$. By Bux-Köhl-Witzel [10], L is of type $FP_{k(\mathbf{G}, S)-1}$ as a $\mathbb{Z}[1/p]\Gamma$ -module. Therefore if Conjecture 2 is true, then $H^1(\Gamma; L)$ is a quotient of $H_{k(\mathbf{G}, S)-1}(\Gamma; D)$, and if $n \neq k(\mathbf{G}, S)$ and $n \neq k(\mathbf{G}, S) - 1$ then

$$H_n(\Gamma; D) \cong H^{k(\mathbf{G}, S)-n}(\Gamma; L).$$

Note that the only dimension of $H^*(\Gamma; L)$ which semiduality would not be able to help determine is dimension 0, but we know $H^0(\Gamma; L) = L$.

1.3. Main result. What we prove in this paper is a first case of Conjecture 2. Namely

Theorem 3. *Conjecture 2 is true if $\text{rank}_{K_v} \mathbf{G} = 1$ for all $v \in S$. In particular, if \mathbf{P} is a proper K -parabolic subgroup of \mathbf{G} , then there is an exact sequence of $\mathbb{Z}[1/p]\mathbf{G}(K)$ -modules*

$$0 \longrightarrow H_c^{k(\mathbf{G}, S)}(X_S; \mathbb{Z}[1/p]) \longrightarrow D \longrightarrow \bigoplus_{z \in (\mathbf{G}/\mathbf{P})(K)} M_z \longrightarrow 0$$

where M_z is an uncountable $\mathbb{Z}[1/p]$ -module for each $z \in (\mathbf{G}/\mathbf{P})(K)$, $M_z \cong M_w$ as $\mathbb{Z}[1/p]$ -modules for any $z, w \in (\mathbf{G}/\mathbf{P})(K)$, and $g(M_z) = M_{gz}$ for all $g \in \mathbf{G}(K)$ and $z \in (\mathbf{G}/\mathbf{P})(K)$.

For example, $\mathbf{SL}_2(\mathbb{F}_p[t])$ is a semiduality group of dimension 1, $\mathbf{SL}_2(\mathbb{F}_p[t, t^{-1}])$ is a semiduality group of dimension 2, and $\mathbf{SL}_2(\mathcal{O}_S)$ is a semiduality group of dimension $|S|$ whose dualizing module incorporates the action of $\mathbf{SL}_2(K)$ on $\mathbb{P}^1(K)$.

Our proof of Theorem 3 is geometric. That is, we will use strongly that, under the hypotheses of Theorem 3, X_S is a product of trees.

1.4. Solvable groups. Let \mathbf{B}_2 be the group of upper triangular matrices of determinant 1. Thus, $\mathbf{B}_2(\mathbb{F}_p[t])$ is commensurable to $\mathbb{F}_p[t]$, and $\mathbf{B}_2(\mathbb{F}_p[t, t^{-1}])$ is commensurable to the lamplighter group $\mathbb{F}_p \wr \mathbb{Z}$. This paper will also show

Theorem 4. $\mathbf{B}_2(\mathcal{O}_S)$ is virtually a $\mathbb{Z}[1/p]$ -semiduality group of dimension $|S|$.

Thus, Solv is a Poincaré duality group, solvable Baumslag-Solitar groups are duality groups, and lamplighter groups with prime order cyclic base are semiduality groups. Notice that these three groups are commensurable respectively to $\mathbf{B}_2(\mathbb{Z}[\sqrt{2}])$, $\mathbf{B}_2(\mathbb{Z}[1/p])$, and $\mathbf{B}_2(\mathbb{F}_p[t, t^{-1}])$.

We also show that certain generalizations of $\mathbf{B}_2(\mathbb{F}_p[t])$ and $\mathbf{B}_2(\mathbb{F}_p[t, t^{-1}])$ are semiduality groups, namely countable sums of finite groups and Diestel-Leader groups, respectively.

1.5. Outline of proof. In Section 2 we'll prove Proposition 1. In Section 3 we'll show how the cohomology of a discrete group with group ring coefficients can be, in some cases, interpreted from the topology of a contractible space on which it acts properly, and perhaps noncocompactly. In Section 4 we'll detail how the groups $\mathbf{G}(\mathcal{O}_S)$ from Theorem 3 act cocompactly on the complement of a pairwise disjoint collection of horoballs in a product of trees, and in Section 5 we'll show that such a complement has trivial compactly supported cohomology in dimension $d - 1$, where d is the number of factors in the product. Section 6 shows that \varprojlim^1 of the compactly supported cohomology of a nested sequence of regular horospheres in a product of trees is torsion-free in dimension

$d - 1$, and the final section of this paper, Section 7, will combine the ingredients collected in earlier sections to prove that certain groups are semiduality groups, including a proof of Theorem 3.

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2. HOMOLOGICAL ALGEBRA

In this section, we let R be a commutative ring and we let Γ be a group that is of type FP_{d-1} over R , but not of type FP_d over R . Further, we assume that the cohomological dimension of Γ over R is d , and we suppose that $H^*(\Gamma; R\Gamma)$ is concentrated in dimension d , where it is a flat R -module. Let $D = H^d(\Gamma; R\Gamma)$

Our goal in this section is to show that Γ has some duality maps, and to provide sufficient conditions for when such maps are injective, surjective, or bijective.

2.1. Bicomplex of resolutions. In the remainder of this section we let M be an $R\Gamma$ -module. and $\cdots \rightarrow Q_1 \rightarrow Q_0 \rightarrow M \rightarrow 0$ be a projective resolution of M by $R\Gamma$ -modules.

We let $0 \rightarrow P_d \rightarrow P_{d-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow R \rightarrow 0$ be an $R\Gamma$ -projective resolution of the trivial $R\Gamma$ -module R , where P_i is finitely generated if $i < d$ and P_d is infinitely generated. Such a resolution exists since Γ is of type FP_{d-1} over R but not of type FP_d .

For each i we let $P_i^* = \text{Hom}_{R\Gamma}(P_i; R\Gamma)$. Since $H^*(\Gamma, R\Gamma)$ is concentrated in dimension d , the sequence

$$0 \rightarrow P_0^* \rightarrow P_1^* \rightarrow \cdots \rightarrow P_{d-1}^* \rightarrow P_d^* \rightarrow D \rightarrow 0$$

is exact. Since P_i is finitely-generated for $i < d$, we see that P_i^* is projective for $i < d$.

We let E be the bicomplex formed from the deleted resolutions $P_{d-\bullet}^*$ and Q_\bullet . That is, we let $E_{p,q} = P_{d-p}^* \otimes_{R\Gamma} Q_q$. We will compute the two associated spectral sequences ${}^{II}E_{p,q}^\bullet$ and ${}^IE_{p,q}^\bullet$, in the notation of [16].

2.2. Second spectral sequence for E . To determine ${}^{II}E_{p,q}^\infty$ by first finding the homology of the rows of $E_{p,q}$, note that each Q_i is flat as it is projective, so the homologies of the rows of $E_{p,q}$ are trivial except along the column $p = 0$ where ${}^{II}E_{0,q}^1 = H_0(P_{d-\bullet}^* \otimes_{R\Gamma} Q_q)$.

Since $P_{d-1}^* \rightarrow P_d^* \rightarrow D \rightarrow 0$ is exact and tensor is right-exact, the sequence

$$P_{d-1}^* \otimes_{R\Gamma} Q_q \rightarrow P_d^* \otimes_{R\Gamma} Q_q \rightarrow D \otimes_{R\Gamma} Q_q \rightarrow 0$$

is exact. Therefore,

$${}^{II}E_{p,q}^1 = \begin{cases} D \otimes_{R\Gamma} Q_q & \text{if } p = 0 \\ 0 & \text{if } p > 0. \end{cases}$$

We see from this that the spectral sequence collapses at the E^2 page. Since Q_\bullet is a deleted projective resolution of M , taking homology of the ${}^{II}E_{1,q}^1$ column computes $\text{Tor}_\bullet^{R\Gamma}(D, M)$, and so

$${}^{II}E_{p,q}^\infty = \begin{cases} \text{Tor}_q^{R\Gamma}(D, M) & \text{if } p = 0 \\ 0 & \text{if } p > 0. \end{cases}$$

Since this spectral sequence converges to the total homology of the bicomplex $E_{p,q} = P_{d-p}^* \otimes_{R\Gamma} Q_q$, we see that

$$H_n(\text{Tot}(E)) = {}^{II}E_{0,n}^\infty = \text{Tor}_n^{R\Gamma}(D, M) = H_n(\Gamma; D \otimes_R M)$$

since D is R -flat.

2.3. First spectral sequence for E . Now we look at the spectral sequence for $E_{p,q}$ that begins with the homology of the columns. Since P_i^* is projective, and hence flat, if $i < d$, we see that all but the column for $p = 0$ is exact. Thus,

$${}^IE_{p,q}^1 = \begin{cases} 0 & \text{if } p > 0 \text{ and } q > 0; \\ H_q(P_d^* \otimes_{R\Gamma} Q_\bullet) & \text{if } p = 0 \text{ and } q > 0; \text{ and} \\ H_0(P_{d-p}^* \otimes_{R\Gamma} Q_\bullet) & \text{if } q = 0. \end{cases}$$

By the definition of Tor , this is the same as

$${}^IE_{p,q}^1 = \begin{cases} 0 & \text{if } p > 0 \text{ and } q > 0; \\ \text{Tor}_q^{R\Gamma}(P_d^*, M) & \text{if } p = 0 \text{ and } q > 0; \text{ and} \\ H_0(P_{d-p}^* \otimes_{R\Gamma} Q_\bullet) & \text{if } q = 0. \end{cases}$$

Since $Q_1 \rightarrow Q_0 \rightarrow M \rightarrow 0$ is exact, the sequence

$$P_{d-p}^* \otimes_{R\Gamma} Q_1 \rightarrow P_{d-p}^* \otimes_{R\Gamma} Q_0 \rightarrow P_{d-p}^* \otimes_{R\Gamma} M \rightarrow 0$$

is also exact, which is to say that we have

$${}^IE_{p,q}^1 = \begin{cases} 0 & \text{if } p > 0 \text{ and } q > 0; \\ \text{Tor}_q^{R\Gamma}(P_d^*, M) & \text{if } p = 0 \text{ and } q > 0; \text{ and} \\ P_{d-p}^* \otimes_{R\Gamma} M & \text{if } q = 0. \end{cases}$$

Now to determine ${}^IE_{p,q}^2$, we need to determine the homology of the complex $P_{d-\bullet}^* \otimes_{R\Gamma} M$. First note that if $p > 0$, then P_{d-p}^* is finitely generated so that $P_{d-p}^* \otimes_{R\Gamma} M = \text{Hom}_{R\Gamma}(P_{d-p}, M)$. Then by definition we have that ${}^IE_{p,0}^2 = H^{d-p}(\Gamma; M)$ if $p > 1$.

Since $P_{d-1}^* \rightarrow P_d^* \rightarrow D \rightarrow 0$ is exact, so is $P_{d-1}^* \otimes_{R\Gamma} M \rightarrow P_d^* \otimes_{R\Gamma} M \rightarrow D \otimes_{R\Gamma} M \rightarrow 0$ which implies that ${}^IE_{0,0}^2 = D \otimes_{R\Gamma} M$. Therefore we have

$${}^IE_{p,q}^2 = \begin{cases} 0 & \text{if } p > 0 \text{ and } q > 0; \\ \text{Tor}_q^{R\Gamma}(P_d^*, M) & \text{if } p = 0 \text{ and } q > 0; \\ D \otimes_{R\Gamma} M & \text{if } p = 0 \text{ and } q = 0; \\ H_1(P_{d-\bullet}^* \otimes_{R\Gamma} M) & \text{if } p = 1 \text{ and } q = 0; \\ H^{d-p}(\Gamma, M) & \text{if } p > 1 \text{ and } q = 0. \end{cases}$$

For $p > 1$ there are maps $d^p : {}^IE_{p,0}^2 \rightarrow {}^IE_{0,p-1}^2$ and ${}^IE_{p,0}^\infty$ and ${}^IE_{0,p-1}^\infty$ are the kernel and cokernel of these maps, respectively. Since ${}^IE_{p,q}^\infty$ converges to the total homology of E , which we had seen above is $H_n(\Gamma; D \otimes_R M)$, for any $n > 0$ we have an exact sequence

$$0 \rightarrow {}^IE_{0,n}^\infty \rightarrow H_n(\Gamma; D \otimes_R M) \rightarrow {}^IE_{n,0}^\infty \rightarrow 0.$$

Lemma 5. ${}^IE_{1,0}^\infty$ is a submodule of $H^{d-1}(\Gamma; M)$. If M is finitely presented then ${}^IE_{1,0}^\infty = H^{d-1}(\Gamma; M)$.

Proof. Recall that ${}^IE_{1,0}^\infty = H_1(P_{d-\bullet}^* \otimes_{R\Gamma} M)$. The following diagram commutes

$$\begin{array}{ccccc} P_d^* \otimes_{R\Gamma} M & \longleftarrow & P_{d-1}^* \otimes_{R\Gamma} M & \longleftarrow & P_{d-2}^* \otimes_{R\Gamma} M \\ \downarrow & & \downarrow & & \downarrow \\ \text{Hom}_{R\Gamma}(P_d, M) & \longleftarrow & \text{Hom}_{R\Gamma}(P_{d-1}, M) & \longleftarrow & \text{Hom}_{R\Gamma}(P_{d-2}, M). \end{array}$$

The homology of the top row is $H_1(P_{d-\bullet}^* \otimes_{R\Gamma} M)$ and the homology of the bottom row is $H^{d-1}(\Gamma; M)$, so the diagram induces a map $H_1(P_{d-\bullet}^* \otimes_{R\Gamma} M) \rightarrow H^{d-1}(\Gamma; M)$. Since P_{d-1} and P_{d-2} are finitely generated, the two vertical arrows on the right are isomorphisms. From there one can check that $H_1(P_{d-\bullet}^* \otimes_{R\Gamma} M) \rightarrow H^{d-1}(\Gamma; M)$ is injective.

If M is finitely presented then the leftmost vertical arrow is an isomorphism (for proof see [4, 1.4.1]), and so the induced map of homology is an isomorphism. \square

Lemma 6. There are maps $\varphi_n^M : H_n(\Gamma; D \otimes_R M) \rightarrow H^{d-n}(\Gamma; M)$ that are injective if $\text{Tor}_n^{R\Gamma}(P_d^*, M) = 0$, and surjective if $\text{Tor}_{n-1}^{R\Gamma}(P_d^*, M) = 0$ and $n > 1$.

Proof. The existence of φ_0^M is standard, following from the fact that the cohomological dimension of Γ is d . (For example, see Lemma 8 below.)

If $n > 0$ we have the sequence

$$(S1) \quad 0 \rightarrow {}^I E_{0,n}^\infty \xrightarrow{f_n} H_n(\Gamma; D \otimes_R M) \xrightarrow{g_n} {}^I E_{n,0}^\infty \rightarrow 0.$$

Suppose $n = 1$. By Lemma 5 we have an inclusion $\iota : {}^I E_{1,0}^\infty \rightarrow H^{d-1}(\Gamma; M)$. Then $\varphi_1^M = \iota \circ g_1$ for g_1 as in sequence S1. The kernel of φ_1^M is ${}^I E_{0,1}^\infty = \text{Tor}_1^{R\Gamma}(P_d^*, M)$, and so if $\text{Tor}_1^{R\Gamma}(P_d^*, M) = 0$ then φ_1^M is injective.

Suppose now $n > 1$. In this case the above calculations show ${}^I E_{n,0}^\infty = H^{d-n}(\Gamma; M)$, so φ_n^M is the map g_n of the sequence S1. Recall that ${}^I E_{0,n}^2 = \text{Tor}_n^{R\Gamma}(P_d^*, M)$. If $\text{Tor}_n^{R\Gamma}(P_d^*, M) = 0$ then ${}^I E_{0,n}^\infty = 0$, which proves injectivity of φ_n^M . If $\text{Tor}_{n-1}^{R\Gamma}(P_d^*, M) = 0$ then the map $d^n : {}^I E_{n,0}^2 \rightarrow {}^I E_{0,n-1}^2$ is trivial. This shows ${}^I E_{n,0}^\infty = H^{d-n}(\Gamma; M)$ and so φ_n^M is surjective. \square

Corollary 7. *If M is flat then the maps $\varphi_n^M : H_n(\Gamma; D \otimes_R M) \rightarrow H^{d-n}(\Gamma; M)$ are injective if $n = 1$ and bijective if $n > 1$.*

Proof. If M is flat then $\text{Tor}_n^{R\Gamma}(P_d^*, M) = 0$ for all $n \geq 1$, so the result follows from Lemma 6. \square

Lemma 8. *If Q_0 is finitely generated, then $H^d(\Gamma; M)$ is a quotient of $H_0(\Gamma; D \otimes_R M)$. If Q_0 and Q_1 are finitely generated, then $H_0(\Gamma; D \otimes_R M) = H^d(\Gamma; M)$.*

Proof. The following diagram commutes. The top row is exact because tensor is right exact. The bottom row is exact by the definition of $H^d(\Gamma, M)$.

$$\begin{array}{ccccccc} 0 & \longleftarrow & D \otimes_{R\Gamma} M & \longleftarrow & P_d^* \otimes_{R\Gamma} M & \longleftarrow & P_{d-1}^* \otimes_{R\Gamma} M \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longleftarrow & H^d(\Gamma, M) & \longleftarrow & \text{Hom}_{R\Gamma}(P_d, M) & \longleftarrow & \text{Hom}_{R\Gamma}(P_{d-1}, M) \end{array}$$

The vertical map on the right is an isomorphism, so $D \otimes_{R\Gamma} M \rightarrow H^d(\Gamma; M)$ is surjective (resp. bijective) if the second vertical map from the right is by the 5-Lemma. Now note that the second vertical map on the right is surjective if M is finitely generated, and bijective if M is finitely presented. (See, for example, [4, 1.4.1]) Last, note that $D \otimes_{R\Gamma} M = H_0(\Gamma; D \otimes_R M)$. \square

Lemma 9. *If $\cdots \rightarrow Q_1 \rightarrow Q_0 \rightarrow M \rightarrow 0$ is a projective resolution of M with Q_{n+1} and Q_n finitely generated, then $\text{Tor}_n^{R\Gamma}(P_d^*, M) = 0$.*

Proof. The following diagram commutes

$$\begin{array}{ccccc}
 P_d^* \otimes_{R\Gamma} Q_{n-1} & \longleftarrow & P_d^* \otimes_{R\Gamma} Q_n & \longleftarrow & P_d^* \otimes_{R\Gamma} Q_{n+1} \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathrm{Hom}_{R\Gamma}(P_d, Q_{n-1}) & \longleftarrow & \mathrm{Hom}_{R\Gamma}(P_d, Q_n) & \longleftarrow & \mathrm{Hom}_{R\Gamma}(P_d, Q_{n+1})
 \end{array}$$

Since Q_{n+1} and Q_n are finitely generated projective, they are finitely presented, so as in the proof of Lemma 8, the two maps on the right are isomorphisms. It follows that the homology of the top row injects into the homology of the bottom row. The homology of the top row is $\mathrm{Tor}_n^{R\Gamma}(P_d^*, M)$ while the bottom row is exact since $\mathrm{Hom}(P_d, -)$ is exact as P_d is projective. \square

Lemma 10. *The maps $\varphi_n^M : H_n(\Gamma; D \otimes_R M) \rightarrow H^{d-n}(\Gamma; M)$ are injective if Q_{n+1} and Q_n are finitely generated, and surjective if Q_n and Q_{n-1} are finitely generated.*

Proof. The $n = 0$ case is Lemma 8.

Suppose $n = 1$. If Q_1 and Q_2 are finitely generated, then by Lemma 9 we have $\mathrm{Tor}_1^{R\Gamma}(P_d^*, M) = 0$ and so φ_1^M is injective by Lemma 6. If Q_0 and Q_1 are finitely generated then M is finitely presented, so φ_1^M is surjective by Lemma 5.

Suppose $n > 1$. If Q_{n+1} and Q_n are finitely generated then $\mathrm{Tor}_n^{R\Gamma}(P_d^*, M) = 0$ by Lemma 9 and so φ_n^M is injective by Lemma 6. If Q_n and Q_{n-1} are finitely generated then $\mathrm{Tor}_{n-1}^{R\Gamma}(P_d^*, M) = 0$ and so φ_n^M is surjective by Lemma 6. \square

In combination, Lemma 10 and Corollary 7 prove Proposition 1 of the introduction.

3. TRANSLATION FROM TOPOLOGY

3.1. Cohomology compactly supported above each level. If a group Γ has a finite Eilenberg-MacLane complex $X = K(\Gamma, 1)$ with universal cover \tilde{X} then for any ring R there is an isomorphism $H^*(\Gamma; R\Gamma) = H_c^*(\tilde{X}; R)$. In this section we provide an alternative topological characterization of $H^*(\Gamma, R\Gamma)$ in the case that X is not finite. Our proof uses standard techniques, which we include for completeness.

Suppose X is a locally finite cell complex with an action by a group Γ , and let $\pi : X \rightarrow \Gamma \backslash X$ denote the quotient map. Let $C^*(X; R)$ denote the cellular cochain complex of X with coefficients in a ring R . Define a subcomplex $C_{c\uparrow}^k(X; R) \leq C^k(X; R)$ to contain cochains $\phi \in C^k(X; R)$ such that for every k -cell $\sigma \in \Gamma \backslash X$ we have $\phi(\tilde{\sigma}) = 0$ for all but finitely many $\tilde{\sigma} \in \pi^{-1}(\sigma)$. It is easy to see that $d(C_{c\uparrow}^k(X; R)) \subseteq C_{c\uparrow}^{k+1}(X; R)$.

Let $H_{c\uparrow}^*(X; R)$ be the cohomology of this complex. We suppress the dependence on the action of Γ from the notation.

Proposition 11. *Suppose X is a locally finite, acyclic cell complex and Γ is a group acting on X with cell stabilizers that are finite and preserve orientation. Then*

$$H^*(\Gamma; R\Gamma) \cong H_{c\uparrow}^*(X; R).$$

Proof. Recall that the equivariant cohomology of the pair (X, Γ) with coefficients in $R\Gamma$ is defined as

$$H_\Gamma^*(X; R\Gamma) = H^*(\Gamma; C^*(X; R\Gamma)).$$

There is an isomorphism (cf. [7, VII.7.3, p173])

$$H^*(\Gamma; R\Gamma) \cong H_\Gamma^*(X; R\Gamma).$$

There is a spectral sequence (cf. [7, p169])

$$E_1^{pq} = H^q(\Gamma; C^p(X; R\Gamma)) \implies H_\Gamma^{p+q}(X; R\Gamma).$$

We will show $H^q(\Gamma; C^p(X; R\Gamma)) = 0$ for all $q > 0$. Let X_p denote the set of p -cells in X and let Σ_p be a set of representatives for $\Gamma \backslash X_p$. Letting Γ_σ denote the stabilizer of $\sigma \in \Sigma_p$, there is a decomposition

$$\begin{aligned} C^p(X; R\Gamma) &= \text{Hom}(C_p(X), R\Gamma) \\ &\cong \prod_{\sigma \in X_p} R\Gamma \\ &\cong \prod_{\sigma \in \Sigma_p} \text{Coind}_{\Gamma_\sigma}^\Gamma(R\Gamma). \end{aligned}$$

Therefore there is a decomposition of cohomology

$$\begin{aligned} H^q(\Gamma; C^p(X; R\Gamma)) &\cong H^q\left(\Gamma; \prod_{\sigma \in \Sigma_p} \text{Coind}_{\Gamma_\sigma}^\Gamma(R\Gamma)\right) \\ &\cong \prod_{\sigma \in \Sigma_p} H^q(\Gamma; \text{Coind}_{\Gamma_\sigma}^\Gamma(R\Gamma)) \end{aligned}$$

Applying Shapiro's lemma yields

$$H^q(\Gamma; C^p(X; R\Gamma)) \cong \prod_{\sigma \in \Sigma_p} H^q(\Gamma_\sigma; R\Gamma).$$

Because Γ_σ is finite, there is an isomorphism of $R\Gamma_\sigma$ -modules $R\Gamma \cong \text{Coind}_{\{1\}}^{\Gamma_\sigma}(\oplus_{\Sigma_p} R)$. Therefore another use of Shapiro's lemma shows that $H^q(\Gamma_\sigma; R\Gamma) = 0$ for $q > 0$. Recall that $H^0(\Gamma_\sigma; R\Gamma) \cong (R\Gamma)^{\Gamma_\sigma}$.

It follows from the above that $H^*(\Gamma; R\Gamma)$ is the cohomology of the cochain complex

$$(1) \quad \prod_{\sigma \in \Sigma_0} (R\Gamma)^{\Gamma_\sigma} \rightarrow \prod_{\sigma \in \Sigma_1} (R\Gamma)^{\Gamma_\sigma} \rightarrow \prod_{\sigma \in \Sigma_2} (R\Gamma)^{\Gamma_\sigma} \rightarrow \dots$$

We will show this chain complex is isomorphic to the chain complex $\{C_{c\uparrow}^k(X, R)\}$. First we compute the coboundary maps of (1). The isomorphism

$$C^p(X; R\Gamma)^\Gamma \rightarrow \prod_{\sigma \in \Sigma_p} (R\Gamma)^{\Gamma_\sigma}$$

sends a map $\phi : C_p(X; R) \rightarrow R\Gamma$ to the function $\xi : \Sigma_p \rightarrow R\Gamma$ defined by $\xi(\sigma) = \phi([\sigma])$ for any p -cell $\sigma \in \Sigma_p$. From Γ -equivariance of ϕ computation shows that the coboundary operator on the complex (1) is given by

$$d\xi(\sigma) = \sum_i r_i \gamma_i \xi(\sigma_i)$$

if $\partial[\sigma] = \sum n_i \gamma_i [\sigma_i]$ for ring elements $r_i \in R$, group elements $\gamma_i \in \Gamma$ and simplices $\sigma_i \in \Sigma_p$.

Define an isomorphism

$$\Theta : \prod_{\sigma \in \Sigma_p} (R\Gamma)^{\Gamma_\sigma} \rightarrow C_{c\uparrow}^p(X, R)$$

as follows: given $\phi \in \prod_{\sigma \in \Sigma_p} (R\Gamma)^{\Gamma_\sigma}$, for any p -simplex ρ in X choose $\sigma \in \Sigma_p$ and $\gamma \in \Gamma$ such that $\rho = \gamma\sigma$ and set

$$\Theta\phi(\rho) = [\phi(\sigma)]_{\gamma^{-1}}.$$

Here $[x]_{\gamma^{-1}}$ denotes the coefficient of $[\gamma^{-1}]$ in the formal sum $x \in R\Gamma$. Note σ is uniquely specified by ρ and γ is unique up to right multiplication by elements of Γ_σ . Because each $\phi(\sigma)$ is Γ_σ -invariant, Θ does not depend on choice of γ . Moreover, any two p -cells in X that belong to the same Γ orbit will correspond to the same cell σ in the above equation. Since there are only finitely many terms in the formal sum $\phi(\sigma)$, the map $\Theta\phi$ is finitely supported above each cell in X . Therefore Θ determines a well-defined homomorphism of Γ -modules.

It is clear that Θ is injective. To see that Θ is surjective, define an inverse Θ^{-1} by setting $[\Theta^{-1}\xi(\sigma)]_\gamma = \xi(\gamma^{-1}\sigma)$. It remains only to see that Θ is compatible with the coboundary maps. Suppose ρ is a $(p+1)$ -cell in X and $\partial[\rho] = \sum_i r_i [\delta_i]$ for ring elements $r_i \in R$ and

p -cells δ_i . For each i , write $\delta_i = \gamma_i \sigma_i$ for $\gamma_i \in \Gamma$ and $\sigma_i \in \Sigma_p$. Then

$$\begin{aligned} d[\Theta\phi](\rho) &= \Theta\phi(\partial\rho) \\ &= \sum_i r_i [\Theta\phi](\delta_i) \\ &= \sum_i r_i [\phi(\sigma_i)]_{\gamma_i^{-1}}. \end{aligned}$$

On the other hand, note that if $\rho = \gamma\sigma$ for $\gamma \in \Gamma$ and $\sigma \in \Sigma_{p+1}$, then

$$\partial[\sigma] = \sum_i r_i (\gamma^{-1}\gamma_i) [\sigma_i].$$

Therefore

$$\begin{aligned} \Theta(d\phi)(\rho) &= [(d\phi)(\sigma)]_{\gamma^{-1}} \\ &= [\phi(\partial\sigma)]_{\gamma^{-1}} \\ &= \left[\sum_i r_i (\gamma^{-1}\gamma_i) \phi(\sigma_i) \right]_{\gamma^{-1}} \\ &= \sum_i r_i [(\gamma^{-1}\gamma_i) \phi(\sigma_i)]_{\gamma^{-1}} \\ &= \sum_i r_i [\phi(\sigma_i)]_{\gamma_i^{-1}} \end{aligned}$$

Thus Θ commutes with the coboundary operators and hence is an isomorphism of chain complexes. This completes the proof. \square

Lemma 12. *Let X and Γ be as in Proposition 11. If G is a locally compact group acting cellularly on X and $\Gamma \leq G$ then $\text{Comm}_G(\Gamma)$ acts on $H_{c\uparrow}^*(X; R)$.*

Proof. Given $\phi \in C_{c\uparrow}^k(X)$ and $g \in \text{Comm}_G(\Gamma)$, define $(g\phi)(\sigma) = \phi(g^{-1}\sigma)$. The condition that $g\phi \in C_{c\uparrow}^k(X)$ is equivalent to the condition that $\text{supp}(g\phi) \cap \Gamma K$ is compact for any compact set $K \subseteq X$, which is equivalent to $\text{supp}(\phi) \cap g^{-1}\Gamma K$ being compact for any compact K . Fix a compact $K \subseteq X$. Because $g \in \text{Comm}_G(\Gamma)$, there is some compact $K' \subseteq X$ such that $g\Gamma K \subseteq \Gamma K'$. Therefore $\text{supp}(\phi) \cap g^{-1}\Gamma K \subseteq \text{supp}(\phi) \cap \Gamma K'$. The latter is compact because $\phi \in C_{c\uparrow}^k(X)$, so $g\phi \in C_{c\uparrow}^k(X)$. This action commutes with coboundary maps, so induces an action on cohomology. \square

3.2. Computing $H_{c\uparrow}^*(X)$. Let X and Γ be as in Proposition 11. Suppose there are subcomplexes $X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots \subseteq X$ such that each X_k is closed and Γ -invariant, each quotient $\Gamma \backslash X_k$ is compact,

and $X = \bigcup_i X_i$. The compactly supported cellular cochain complexes $C_c^*(X_n; R)$ form a codirected system under the restriction maps $r_{i,j}^*$ induced by inclusions $r_{i,j} : X_i \rightarrow X_j$ for $i < j$.

The chain complex $C_{c\uparrow}^k(X; R)$ is the inverse limit of the system of chain complexes $C_c^k(X_n; R)$ and each restriction map $r_{i,j}$ is surjective on the chain level. It follows (see for example the “Variant” following [18, 3.5.8, p84]) that for any k there is a short exact sequence of cohomology

$$(2) \quad 0 \rightarrow \varprojlim^1 H_c^{k-1}(X_n; R) \rightarrow H_{c\uparrow}^k(X; R) \rightarrow \varprojlim H_c^k(X_n; R) \rightarrow 0.$$

Recall that $\varprojlim^1 H_c^{k-1}(X_n; R)$ is the cokernel of the map

$$\Delta : \prod_{n=0}^{\infty} H_c^{k-1}(X_n; R) \rightarrow \prod_{n=0}^{\infty} H_c^{k-1}(X_n; R)$$

defined by

$$\Delta(x_0, x_1, x_2, \dots) = (x_0 - r_{0,1}(x_1), x_1 - r_{1,2}(x_2), \dots).$$

As a straightforward application of the short exact sequence (2) we have:

Proposition 13. *Suppose X is a locally finite cell complex with an action of a group Γ . Suppose X is the union of an increasing sequence of Γ -invariant subcomplexes X_i each with cocompact Γ action. If there is some integer d such that $H_c^*(X_n; R)$ is concentrated in dimension d for all n then $H_{c\uparrow}^*(X; R)$ is concentrated in dimensions d and $d+1$. In particular, if X is d -dimensional then*

$$H_{c\uparrow}^k(X; R) = \begin{cases} \varprojlim_n H_c^d(X_n; R) & k = d \\ 0 & k \neq d. \end{cases}$$

4. STATEMENT OF REDUCTION THEORY

In this section we’ll review the necessary results needed from reduction theory for our proof of Theorem 3. The results in this section are not new, and can be derived from Behr [2] and Harder [14], although there are some minor differences between our treatment of reduction theory here and other versions already existing in the literature. A point of difference in the proof of our formulation of these results compared with formulations in other papers, is that we’ll use the reduction theory from Bestvina-Eskin-Wortman [3] as an input, which has the advantage, though not directly applied in this paper, of being equally applicable to arithmetic groups defined with respect to a number field. See also Bux-Wortman [12] and Bux-Köhl-Witzel [10].

4.1. Algebraic form of reduction theory. In this section and the next we assume that K is a global field with a ring of S -integers $\mathcal{O}_S \leq K$ and that \mathbf{G} is a noncommutative, absolutely almost simple, K -isotropic, K -group with $\text{rank}_{K_v}(\mathbf{G}) = 1$ for all $v \in S$.

Let \mathbf{P} be a proper K -parabolic subgroup of \mathbf{G} . Let \mathbf{A} be a maximal K -split torus in \mathbf{P} .

From the root system for (\mathbf{G}, \mathbf{A}) , we denote the simple root for the positive roots with respect to \mathbf{P} by α_0 .

We let $\mathbf{Z}_{\mathbf{G}}(\mathbf{A})$ be the centralizer of \mathbf{A} in \mathbf{G} so that $\mathbf{Z}_{\mathbf{G}}(\mathbf{A}) = \mathbf{M}\mathbf{A}$ where \mathbf{M} is a reductive K -group with K -anisotropic center. We let \mathbf{U} be the unipotent radical of \mathbf{P} , so that $\mathbf{P} = \mathbf{U}\mathbf{M}\mathbf{A}$. The Levi subgroup $\mathbf{M}\mathbf{A}$ normalizes the unipotent radical \mathbf{U} , and elements of \mathbf{A} commute with those of \mathbf{M} .

We denote the product over S of local points of a K -group by “unbolding”, so that, for example,

$$G = \prod_{v \in S} \mathbf{G}(K_v)$$

We let \mathcal{P} be the set of proper K -parabolic subgroups of \mathbf{G} . If $\mathbf{Q} \in \mathcal{P}$, then \mathbf{Q} is conjugate in $\mathbf{G}(K)$ to \mathbf{P} . We let

$$\Lambda_{\mathbf{Q}} = \{ \gamma f \in \mathbf{G}(\mathcal{O}_S)F \mid (\gamma f)\mathbf{P}(\gamma f)^{-1} = \mathbf{Q} \}$$

where $F \subseteq \mathbf{G}(K)$ is a finite set of coset representatives for $\mathbf{G}(\mathcal{O}_S) \backslash \mathbf{G}(K) / \mathbf{P}(K)$. Note that if $\gamma_1 f_1, \gamma_2 f_2 \in \Lambda_{\mathbf{Q}}$, then $f_1 = f_2$.

Given any $a = (a_v)_{v \in S} \in A$, we let

$$|\alpha_0(a)| = \prod_{v \in S} |\alpha_0(a_v)|_v$$

where $|\cdot|_v$ is the v -adic norm on K_v .

Given any $t > 0$, we let

$$A^+(t) = \{ a \in A \mid |\alpha_0(a)| \geq t \}$$

and for $t > 0$, we let

$$R_{\mathbf{Q}}(t) = \Lambda_{\mathbf{Q}} \mathbf{U} \mathbf{M} \mathbf{A}^+(t)$$

The following is a special case of Proposition 9 from Bestvina-Eskin-Wortman [3].

Proposition 14. *There exists a bounded set $B_0 \subseteq G$, and given any $N_0 \geq 0$, there exists $t_0 > 1$ and a second bounded set $B_1 \subseteq G$ such that*

- (i) $G = \bigcup_{\mathbf{Q} \in \mathcal{P}} R_{\mathbf{Q}}(1)B_0$;
- (ii) *if $\mathbf{Q}, \mathbf{Q}' \in \mathcal{P}$ and $\mathbf{Q} \neq \mathbf{Q}'$, then the distance between $R_{\mathbf{Q}}(t_0)B_0$ and $R_{\mathbf{Q}'}(t_0)B_0$ is at least N_0 ;*

- (iii) $\mathbf{G}(\mathcal{O}_S) \cap R_{\mathbf{Q}}(t_0)B_0 = \emptyset$; and
- (iv) $G - (\bigcup_{\mathbf{Q} \in \mathcal{P}} R_{\mathbf{Q}}(2t_0)B_0)$ is contained in $\mathbf{G}(\mathcal{O}_S)B_1$.

4.2. Geometric form of reduction theory. We will now reformulate Proposition 14 into a more explicit geometric statement in the form of Proposition 18 below.

For $v \in S$, we let X_v be the Euclidean building for $\mathbf{G}(K_v)$, so that X_v is a tree. We let $X_S = \prod_{v \in S} X_v$.

Let $\Sigma_v \subseteq X_v$ be the geodesic that $\mathbf{A}(K_v)$ acts on by translations. We let $\Sigma_S = \prod_{v \in S} \Sigma_v$, so that Σ_S is isometric to the Euclidean space $\mathbb{R}^{|S|}$.

We define a linear functional $\widehat{\alpha}_0 : \Sigma_S \rightarrow \mathbb{R}$ by associating a basepoint $e \in \Sigma_S$ with the origin as follows:

$$\widehat{\alpha}_0(ae) = \log_p |\alpha_0(a)|$$

for $a \in A$. The action of A on e factors through $\mathbb{Z}^{|S|}$, where $\widehat{\alpha}_0$ is linear, so $\widehat{\alpha}_0$ extends to a functional on all of Σ_S . Furthermore, $\widehat{\alpha}_0$ is nonzero since there is some a with $|\alpha_0(a)| \neq 1$.

For any $r \in \mathbb{R}$, we let $\Sigma_{S,r} \subseteq \Sigma_S$ be

$$\Sigma_{S,r} = \{ x \in \Sigma_S \mid \widehat{\alpha}_0(x) = r \}$$

Thus, $\Sigma_{S,r}$ is a hyperplane in Σ_S that is a finite Hausdorff distance from $\mathbf{A}(\mathcal{O}_S)e \subseteq \Sigma_{S,0}$.

Note that $\Sigma_{S,r}$ is not singular if $|S| > 1$. That is to say, the projection of $\Sigma_{S,r}$ to each Σ_v is surjective if $|S| > 1$. Indeed, to verify this claim observe that if $v \in S$, then $\mathbf{A}(\mathcal{O}_S)$ has dense projection to $\mathbf{A}(K_v)$, and thus acts cocompactly on Σ_v .

Now consider the geodesics Σ_v to be parameterized as unit speed $\Sigma_v : \mathbb{R} \rightarrow X_v$ with $\Sigma_v(\infty) = \mathbf{P}$. From our description of $\widehat{\alpha}_0 : \Sigma_S \rightarrow \mathbb{R}$, we see that there are positive real numbers λ_v such that if $\rho_S : \mathbb{R} \rightarrow X_S$ is given by $\rho_S(t) = (\Sigma_v(\lambda_v t))_{v \in S}$, and if $\beta_{\rho_S} : X_S \rightarrow \mathbb{R}$ is the Busemann function for ρ_S – that is if $x \in X_S$, and d is the distance function on X_S , then

$$\beta_{\rho_S}(x) = \lim_{t \rightarrow \infty} (t - d(x, \rho_S(t)))$$

– then β_{ρ_S} restricted to Σ_S is exactly $\widehat{\alpha}_0$.

Let

$$\begin{aligned} \Sigma_{S,r}^+ &= \{ x \in \Sigma_S \mid \widehat{\alpha}_0(x) \geq r \} \\ &= \{ x \in \Sigma_S \mid \beta_{\rho_S}(x) \geq r \} \end{aligned}$$

so that $\Sigma_{S,r}^+$ is a half space in Σ_S whose boundary equals $\Sigma_{S,r}$.

We let

$$B_{\mathbf{P},S,r} = \{ x \in X_S \mid \beta_{\rho_S}(x) \geq r \}$$

and

$$Y_{\mathbf{P},S,r} = \{ x \in X_S \mid \beta_{\rho_S}(x) = r \}$$

Lemma 15. $B_{\mathbf{P},S,r} = UM\Sigma_{S,r}^+$ and $Y_{\mathbf{P},S,r} = UM\Sigma_{S,r}$.

Proof. \mathbf{M} is contained in both \mathbf{P} and the parabolic group opposite to \mathbf{P} with respect to \mathbf{A} . Also note that $\mathbf{M}(K_v)$ is compact for all $v \in S$. It follows that $\mathbf{M}(K_v)$ fixes Σ_v pointwise, and thus that M fixes Σ_S pointwise. Therefore, $UM\Sigma_{S,r}^+ = U\Sigma_{S,r}^+$.

Elements of $\mathbf{U}(K_v)$ fix unbounded positive rays in Σ_v , thus elements of U fix pointwise a subray of ρ_S , thus β_{ρ_S} is invariant under multiplication by U . Therefore $UB_{\mathbf{P},S,r} = B_{\mathbf{P},S,r}$, so $UM\Sigma_{S,r}^+ \subseteq B_{\mathbf{P},S,r}$ follows from $\Sigma_{S,r}^+ \subseteq B_{\mathbf{P},S,r}$.

To see that $B_{\mathbf{P},S,r} \subseteq U\Sigma_{S,r}^+$, let $x \in B_{\mathbf{P},S,r}$. Since $X_v = \mathbf{U}(K_v)\Sigma_v$, we see that $x = u(x_v)_{v \in S}$ for some $u \in U$ and $x_v \in \Sigma_v$. Thus, $x \in U\Sigma_{S,r}^+$, again, since β_{ρ_S} is invariant under multiplication by U .

That $Y_{\mathbf{P},S,r} = UM\Sigma_{S,r}$ follows similarly. \square

Given $t \in \mathbb{R}$, let $r_t \in \mathbb{R}$ be the supremum of all $r \in \mathbb{R}$ such that $\Sigma_{S,r}^+$ contains $A^+(t)e$. Notice that there is some $C > 0$, independent of t , such that the Hausdorff distance between $A^+(t)e$ and Σ_{S,r_t}^+ is bounded by C . Notice also that $t \mapsto r_t$ is an increasing function.

Lemma 16. *The Hausdorff distance between $UMA^+(t)e$ and $B_{\mathbf{P},S,r_t}$ is bounded independent of t .*

Proof. Because the Hausdorff distance between $A^+(t)e$ and Σ_{S,r_t}^+ is bounded, the Hausdorff distance between $UMA^+(t)e$ and $UM\Sigma_{S,r_t}^+ = B_{\mathbf{P},S,r_t}$ is bounded. \square

Lemma 17. *Let $\mathbf{Q} \in \mathcal{P}$. If $\gamma \in \mathbf{G}(\mathcal{O}_S)$ and $f \in F$ are such that $\gamma f \in \Lambda_{\mathbf{Q}}$, then for any r , we have $\mathbf{Q}(\mathcal{O}_S)\gamma f B_{\mathbf{P},S,r} = \gamma f B_{\mathbf{P},S,r}$.*

Proof. Note that as $B_{\mathbf{P},S,r}$ is given by the Busemann function for ρ_S , $\gamma f B_{\mathbf{P},S,r}$ is given by the Busemann function for $\gamma f \rho_S$.

Since $\gamma f \in \Lambda_{\mathbf{Q}}$, the positive end of each $\gamma f \Sigma_v$ limits to \mathbf{Q} . Thus, if $g \in \mathbf{Q}(K_v)$, then $\gamma f \Sigma_v$ and $g \gamma f \Sigma_v$ intersect in a positive ray. Hence, if $g \in \mathbf{Q}$, then $g \gamma f \rho_S$ is a finite Hausdorff distance from $\gamma f \rho_S$ and $g \gamma f B_{\mathbf{P},S,r} = \gamma f B_{\mathbf{P},S,r_g}$ for some $r_g \in \mathbb{R}$. By replacing g with its inverse, we may assume that $r_g \geq r$.

We may assume that the set B_0 from Proposition 14 is a sufficiently large neighborhood of $1 \in G$, independent of g , so that, in particular there is a set $B' \subseteq B_0$ containing the point stabilizer of 1 and such that

$B'B' \subseteq B_0$, and by the previous lemma, such that $UMA^+(t)B'e$ contains every vertex of $B_{\mathbf{P},S,r_t}$.

Let t_0 be as in Proposition 14. If $r_g \neq r$, then for sufficiently large n we have $g^n e \in \gamma f B_{\mathbf{P},S,r_{t_0}}$. Hence, $g^n e \in \gamma f UMA^+(t_0)B'e$. Therefore, $g^n \in \gamma f UMA^+(t_0)B'B' \subseteq R_{\mathbf{Q}}(t_0)B_0$. We conclude, by Proposition 14 part (iii), that $g \notin \mathbf{G}(\mathcal{O}_S)$. \square

If $\mathbf{Q} \in \mathcal{P}$, we define

$$B_{\mathbf{Q},S,r} = \gamma f B_{\mathbf{P},S,r}$$

for any $\gamma f \in \Lambda_{\mathbf{Q}}$. This is well-defined by the previous lemma, and we also see that $\mathbf{Q}(\mathcal{O}_S)B_{\mathbf{Q},S,r} = B_{\mathbf{Q},S,r}$ and that the Hausdorff distance between $R_{\mathbf{Q}}(t)B_0e$ and $B_{\mathbf{Q},S,r_t}$ is bounded independent of t or \mathbf{Q} . Using this and that the orbit map $G \rightarrow Ge \subseteq X_S$ is proper, we deduce from Proposition 14 the following

Proposition 18. *There exists some $r_0 \in \mathbb{R}$, and given any $N \geq 0$, there is some $r_1 > r_0$ such that*

- (i) $\bigcup_{\mathbf{Q} \in \mathcal{P}} B_{\mathbf{Q},S,r_0} = X_S$;
- (ii) if $\mathbf{Q}, \mathbf{Q}' \in \mathcal{P}$ and $\mathbf{Q} \neq \mathbf{Q}'$, then the distance between $B_{\mathbf{Q},S,r_1}$ and $B_{\mathbf{Q}',S,r_1}$ is at least N ;
- (iii) $\mathbf{G}(\mathcal{O}_S)e \cap B_{\mathbf{Q},S,r_1} = \emptyset$; and
- (iv) $X_S - (\bigcup_{\mathbf{Q} \in \mathcal{P}} B_{\mathbf{Q},S,r_1})$ is a finite Hausdorff distance from $\mathbf{G}(\mathcal{O}_S)e$.

For any $r \in \mathbb{R}$, we let $X_{S,r}$ be the closure in X_S of $X_S - (\bigcup_{\mathbf{Q} \in \mathcal{P}} B_{\mathbf{Q},S,r})$.

Lemma 19. *For $r \gg 0$, $\mathbf{G}(\mathcal{O}_S)$ acts properly and cocompactly on $X_{S,r}$.*

Proof. Let $\gamma \in \mathbf{G}(\mathcal{O}_S)$. Then $\gamma B_{\mathbf{Q},S,r} = B_{\gamma \mathbf{Q} \gamma^{-1},S,r}$ so $\mathbf{G}(\mathcal{O}_S)$ acts on $\bigcup_{\mathbf{Q} \in \mathcal{P}} B_{\mathbf{Q},S,r}$ and thus on $X_{S,r}$.

Since $\mathbf{G}(\mathcal{O}_S)$ acts properly on X_S , it acts properly on $X_{S,r}$.

That $\mathbf{G}(\mathcal{O}_S)$ acts cocompactly on $X_{S,r}$ follows from (iv) of Proposition 18. \square

5. COHOMOLOGY OF THE COMPLEMENT OF DISJOINT HOROBALLS

In this section, we'll examine the cohomology of subspaces of X_S that include spaces of the form $X_{S,r}$, but are slightly more general in that we will allow ourselves to set the height of each horoball individually, rather than use a single parameter to define the height of all horoballs simultaneously. Precisely, for any tuple $(r_{\mathbf{Q}})_{\mathbf{Q} \in \mathcal{P}} \in (\mathbb{R} \cup \{\infty\})^{\mathcal{P}}$, we let $X_{S,(r_{\mathbf{Q}})}$ be the closure of $X_S - (\bigcup_{\mathbf{Q} \in \mathcal{P}} B_{\mathbf{Q},S,r_{\mathbf{Q}}})$ in X_S , where $B_{\mathbf{Q},S,\infty}$ is taken to be the empty set.

We shall call a tuple $(r_{\mathbf{Q}})_{\mathbf{Q} \in \mathcal{P}} \in (\mathbb{R} \cup \{\infty\})^{\mathcal{P}}$ *sufficiently large* if the resulting sets $B_{\mathbf{Q}, S, r_{\mathbf{Q}}}$ are pairwise disjoint, and if their pairwise distance is bounded below by a constant that is sufficiently large. It's known that if $(r_{\mathbf{Q}})_{\mathbf{Q} \in \mathcal{P}}$ is sufficiently large then $X_{S, (r_{\mathbf{Q}})}$ is $(|S| - 2)$ -connected but not $(|S| - 1)$ -connected (see Stuhler [17], Bux-Wortman [12], and Bux-Köhl-Witzel [10]), but these topological properties are not directly relevant to this paper. What we require in this paper, and what we will prove in this section, is that $H_c^k(X_{S, (r_{\mathbf{Q}})}) = 0$ if $k \leq |S| - 1$ and $(r_{\mathbf{Q}})_{\mathbf{Q} \in \mathcal{P}}$ is sufficiently large. (See Proposition 33 below.) We will begin an inductive proof of this claim by observing that the claim is true when $|S| = 1$.

Lemma 20. *If $|S| = 1$, and if $(r_{\mathbf{Q}})_{\mathbf{Q} \in \mathcal{P}}$ is sufficiently large, then the group $H_c^0(X_{S, (r_{\mathbf{Q}})})$ is trivial, where coefficients are in a ring R .*

Proof. In this case, X_S is a tree, and we want to show that the components of $X_{S, (r_{\mathbf{Q}})}$ are unbounded. Indeed, choose an edge $e_0 \in X_{S, (r_{\mathbf{Q}})}$. Because $(r_{\mathbf{Q}})_{\mathbf{Q}}$ is sufficiently large, there is an adjacent edge $e_1 \in X_{S, (r_{\mathbf{Q}})}$, and we can continue in this fashion to create an path of infinite length in $X_{S, (r_{\mathbf{Q}})}$ that begins with e_0 . \square

Our proof of Proposition 33 will include an investigation of spaces that are quite similar to the space $X_{S, (r_{\mathbf{Q}})}$. Precisely, for any $(r_{\mathbf{Q}})_{\mathbf{Q} \in \mathcal{P}}$, let $W_{S, (r_{\mathbf{Q}})}$ be the subcomplex of X_S consisting of all cells of X_S that are contained in $X_{S, (r_{\mathbf{Q}})}$. To see that there isn't much difference between $X_{S, (r_{\mathbf{Q}})}$ and $W_{S, (r_{\mathbf{Q}})}$ we have

Lemma 21. *If the tuple $(r_{\mathbf{Q}})_{\mathbf{Q} \in \mathcal{P}}$ is sufficiently large, then there is a proper homotopy equivalence between $W_{S, (r_{\mathbf{Q}})}$ and $X_{S, (r_{\mathbf{Q}})}$.*

Proof. The proof is an observation through Morse theory. Suppose that $\mathfrak{C} \subseteq X_S$ is a chamber that intersects $X_{S, (r_{\mathbf{Q}})}$ nontrivially, but is not contained in $X_{S, (r_{\mathbf{Q}})}$, and thus is not contained in $W_{S, (r_{\mathbf{Q}})}$.

Because $(r_{\mathbf{Q}})_{\mathbf{Q} \in \mathcal{P}}$ is sufficiently large, \mathfrak{C} intersects $B_{\mathbf{Q}, S, r_{\mathbf{Q}}}$ for a unique \mathbf{Q} . Recall that $B_{\mathbf{Q}, S, r_{\mathbf{Q}}}$ is defined as the inverse image of a positive ray with respect to the Busemann function $\beta_{\gamma f \rho_S} : X_S \rightarrow \mathbb{R}$ associated to the geodesic $\gamma f \rho_S \subseteq X_S$ where $\gamma f \in \Lambda_{\mathbf{Q}}$.

Let $(x_v)_{v \in S}$ be the maximum point of \mathfrak{C} with respect to $\beta_{\gamma f \rho_S}$. Let \mathcal{L}_v be the descending link of x_v in the tree X_v with respect to $\beta_{\gamma f \rho_S} : X_v \rightarrow \mathbb{R}$. We let \mathcal{C}_v be the cone on \mathcal{L}_v taken at x_v in the tree X_v .

For $T \subseteq S$, we let $K_T = \prod_{v \in T} \mathcal{C}_v \times \prod_{v \notin T} \mathcal{L}_v$.

Now we are assuming that $(x_v)_{v \in S} \notin X_{S, (r_{\mathbf{Q}})}$, and note that $K_S - (x_v)_{v \in S}$ deformation retracts onto $\cup_{v \in S} K_{S-v}$ in such a way that the homotopy is nonincreasing with respect to $\beta_{\gamma f \rho_S}$. Note further that

the maximum points in any K_{S-v_0} with respect to $\beta_{\gamma f \rho_S}$ are points of the form $(y_v)_{v \in S}$ where $y_v = x_v$ if $v \neq v_0$, and if these points are not in $X_{S, (r_Q)}$, then we can further retract K_{S-v_0} minus these maximums onto $\cup_{v \in S-v_0} K_{S-\{v_0, v\}}$. We continue in this fashion until all of K_S has been retracted onto some union of K_T with $K_T \subseteq X_{S, (r_Q)}$. \square

In particular, the previous two lemmas show that

Lemma 22. *If $|S| = 1$, and if $(r_Q)_{Q \in \mathcal{P}}$ is sufficiently large, then $H_c^0(W_{S, (r_Q)}) = 0$.*

This lemma will serve as the base step for our inductive proof that $H_c^k(W_{S, (r_Q)}) = 0$ if $k \leq |S| - 1$ and $(r_Q)_{Q \in \mathcal{P}}$ is sufficiently large, which implies that $H_c^k(X_{S, (r_Q)}) = 0$ if $k \leq |S| - 1$ and $(r_Q)_{Q \in \mathcal{P}}$ is sufficiently large.

5.1. Proper products. Now we will focus on the case when $|S| \geq 2$. We choose some $w \in S$ and let $\pi_w : X_S \rightarrow X_w$ be the projection.

Note that by definition of $W_{S, (r_Q)}$, if e is an edge in X_w , and if e° is the interior of e , then $\pi_w|_{W_{S, (r_Q)}} : W_{S, (r_Q)} \rightarrow X_w$ has $\pi_w^{-1}(e^\circ) = e^\circ \times Z_e$ for some complex $Z_e \subseteq X_{S-w}$. Our inductive proof in the remainder of this section is aided by observing that the fibers π_w restricted to one of these “ W spaces” is another “ W space”.

Lemma 23. *For any edge $e \subseteq X_w$, $Z_e = W_{S-w, (s_Q^e)}$ for some tuple $(s_Q^e)_{Q \in \mathcal{P}}$. Furthermore, by choosing $(r_Q)_{Q \in \mathcal{P}}$ sufficiently large we may assume that $(s_Q^e)_{Q \in \mathcal{P}}$ is sufficiently large for each edge $e \subseteq X_w$.*

Proof. Let $x_w \in X_w$ be the endpoint of e that maximizes $\beta_{\gamma f \Sigma_w}$ for $\gamma f \in \Lambda_Q$. Then a cell $\mathfrak{F} \subseteq X_{S-w}$ is contained in Z_e exactly if

$$\beta_{\gamma f \rho_S}(e \times \mathfrak{F}) \leq r_Q$$

which is equivalent to

$$\beta_{\gamma f \rho_S}(x_w \times \mathfrak{F}) \leq r_Q$$

and thus to $\beta_{\gamma f \rho_{(S-w)}}(\mathfrak{F}) \leq s_Q^e$ for some s_Q^e depending $\beta_{\gamma f \Sigma_w}(x_w)$, and thus on e . \square

Lemma 24. *Let $\gamma f \in \Lambda_Q$. If $e_1, e_2 \in X_w$ are edges, and if the maximum of $\beta_{\gamma f \Sigma_w}(e_1)$ is greater than or equal to the maximum of $\beta_{\gamma f \Sigma_w}(e_2)$, then $s_Q^{e_1} \leq s_Q^{e_2}$. If $\beta_{\gamma f \Sigma_w}(e_1) = \beta_{\gamma f \Sigma_w}(e_2)$, then $s_Q^{e_1} = s_Q^{e_2}$.*

Proof. Let $\chi_w \subseteq X_w$ be a geodesic that limits to Q , and suppose that $e_1 \subseteq \chi_w$.

First assume that that $e_2 \subseteq \chi_w$. Then since $\beta_{\gamma f \Sigma_w}(e_2) \leq \beta_{\gamma f \Sigma_w}(e_1)$ we see that $s_{\mathbf{Q}}^{e_2} \geq s_{\mathbf{Q}}^{e_1}$ as desired.

If e_2 is not contained in χ_w , then there is some $u \in \mathbf{U}(K_w)$ such that $u\chi_w$ does contain e_2 . The result follows from the above as $\beta_{\gamma f \rho_{S-w}}$ and $\beta_{\gamma f \Sigma_w}$ are invariant by translations of $\mathbf{U}(K_w)$. \square

Given a vertex $y \in X_w$, we let E_y be the set of edges in X_w that contain y . Then the previous lemma produces

Lemma 25. *For any vertex $y \in X_w$, and any parabolic $\mathbf{Q} \in \mathcal{P}$, either $\{s_{\mathbf{Q}}^e\}_{e \in E_y}$ contains a single value, or else $\{s_{\mathbf{Q}}^e\}_{e \in E_y}$ contains exactly two values, and the minimum value is realized by a unique edge in E_y .*

Proof. For $\gamma f \in \Lambda_{\mathbf{Q}}$, observe that there is a unique edge containing y that maximizes the Busemann function $\beta_{\gamma f \Sigma_w}$, and that the remaining edges minimize $\beta_{\gamma f \Sigma_w}$. \square

In what follows, we'll denote the unique edge in E_y from the proof of the previous lemma as $e(y, \mathbf{Q})$. Thus if $e, \epsilon \in E_y$, then $s_{\mathbf{Q}}^e \leq s_{\mathbf{Q}}^\epsilon$ if $e = e(y, \mathbf{Q})$, and $s_{\mathbf{Q}}^e = s_{\mathbf{Q}}^\epsilon$ if $e, \epsilon \neq e(y, \mathbf{Q})$.

We will need one more related observation about the fibers of π_w in the form of the following

Lemma 26. *If there is a vertex $y \in X_w$, and a cell $\mathfrak{F} \subseteq X_{S-w}$ such that $y \times \mathfrak{F} \subseteq W_{S, (r_{\mathbf{Q}})}$, then $e \times \mathfrak{F} \subseteq W_{S, (r_{\mathbf{Q}})}$ for each $e \in E_y - e(y, \mathbf{Q})$.*

Proof. Let $\gamma f \in \Lambda_{\mathbf{Q}}$. Since $y \times \mathfrak{F} \subseteq W_{S, (r_{\mathbf{Q}})}$, the values of $\beta_{\gamma f \rho_S}(y \times \mathfrak{F})$ are bounded above by $r_{\mathbf{Q}}$. Since y maximizes the values of e under $\beta_{\gamma f \Sigma_w}$, the values of $\beta_{\gamma f \rho_S}(e \times \mathfrak{F})$ are bounded above by $r_{\mathbf{Q}}$ as well. That is, $e \times \mathfrak{F} \subseteq W_{S, (r_{\mathbf{Q}})}$. \square

5.2. Cover by fibers. Having collected some information about the fibers of $\pi_w|_{W_{S, (r_{\mathbf{Q}})}}$, we will now use a collection of fibers to create a cover for $W_{S, (r_{\mathbf{Q}})}$.

For any edge $e \subseteq X_w$, let $F_e = e \times W_{S-w, (s_{\mathbf{Q}}^e)}$ where $W_{S-w, (s_{\mathbf{Q}}^e)}$ is as in Lemma 23.

Lemma 27. *The collection $\{F_e\}$ taken over all edges $e \subseteq X_w$ is a cover for $W_{S, (r_{\mathbf{Q}})}$.*

Proof. Suppose $\sigma \times \mathfrak{F}$ is a cell in $W_{S, (r_{\mathbf{Q}})}$, where σ is a cell in an edge $e \subseteq X_w$ and \mathfrak{F} is a cell in X_{S-w} .

If $\sigma = e$, then $\sigma \times \mathfrak{F} \subseteq F_e$ by Lemma 23. If σ is a vertex of e , say y , then by Lemma 26, there is some e' such that $y \times \mathfrak{F} \subseteq e' \times \mathfrak{F} \subseteq F_{e'}$. \square

For any vertex $y \in X_w$, let $F_y = \cup_{e \in E_y} F_e$. Note that there is a proper homotopy equivalence between F_y and

$$\cup_{y \in e} W_{S-w, (s_{\mathbf{Q}}^e)} = W_{S-w, (\max_{e \in E_y} \{s_{\mathbf{Q}}^e\})}$$

given by retracting the star of y in X_w to the point y .

Further, if $e \in E_y$, then the inclusion $F_e \rightarrow F_y$, after proper homotopy equivalence, is the inclusion $W_{S-w, (s_{\mathbf{Q}}^e)} \rightarrow W_{S-w, (\max_{e \in E_y} \{s_{\mathbf{Q}}^e\})}$. In particular, if $e \in E_y$, then we can, and we shall, identify the map induced by inclusion

$$\rho_{y,e} : H_c^{|S|-1}(F_y) \rightarrow H_c^{|S|-1}(F_e)$$

with the map

$$\rho_{y,e} : H_c^{|S|-1}(W_{S-w, (\max_{e \in E_y} \{s_{\mathbf{Q}}^e\})}) \rightarrow H_c^{|S|-1}(W_{S-w, (s_{\mathbf{Q}}^e)})$$

5.3. Maps between the cohomology of the fibers. For an edge $e \subseteq X_w$, and a parabolic group $\mathbf{R} \in \mathcal{P}$, we let $\mathcal{S}_{e,\mathbf{R}} \subseteq W_{S-w, (s_{\mathbf{Q}}^e)}$ be the complex comprised of all cells $\mathcal{F} \subseteq W_{S-w, (s_{\mathbf{Q}}^e)}$ such that there is a cell $\mathcal{G} \subseteq X_{S-w}$ containing \mathcal{F} with $\beta_{\gamma f \rho_{(S-w)}}(\mathcal{G}) \not\leq s_{\mathbf{R}}^e$ where $\gamma f \in \Lambda_{\mathbf{R}}$. Thus we may informally think of the boundary of $W_{S-w, (s_{\mathbf{Q}}^e)}$ as $\coprod_{\mathbf{Q} \in \mathcal{P}} \mathcal{S}_{e,\mathbf{Q}}$.

Let $y \in X_w$ be a vertex, $e \in E_y$, and $\mathbf{R} \in \mathcal{P}$. We define $\mathcal{J}_{y,e,\mathbf{R}}$ to be the union of cells $\mathcal{F} \subseteq W_{S-w, (\max_{e \in E_y} \{s_{\mathbf{Q}}^e\})}$ such that the maximum value of $\beta_{\gamma f \rho_{(S-w)}}(\mathcal{F})$ is greater than $s_{\mathbf{R}}^e$ for $\gamma f \in \Lambda_{\mathbf{R}}$. Notice that if $\mathcal{J}_{y,e,\mathbf{R}} \neq \emptyset$, then $s_{\mathbf{R}}^e < s_{\mathbf{R}}^\epsilon$ for some $\epsilon \in E_y$, which, by Lemma 25, implies that $e = e(y, \mathbf{R})$.

Lemma 28. *If $y \in X_w$, $e \in E_y$, and $\mathbf{R} \in \mathcal{P}$, then $H_c^{|S|-1}(\mathcal{J}_{y,e,\mathbf{R}}) = 0$.*

Proof. We may assume that $\mathcal{J}_{y,e,\mathbf{R}} \neq \emptyset$, so that $s_{\mathbf{R}}^e < s_{\mathbf{R}}^\epsilon$ for $\epsilon \in E_y - e$. Then $\mathcal{J}_{y,e,\mathbf{R}}$ is the complex of cells \mathcal{F} such that the maximum value of $\beta_{\gamma f \rho_{(S-w)}}(\mathcal{F})$ is greater than $s_{\mathbf{R}}^e$ but bounded above by $s_{\mathbf{R}}^\epsilon$.

Let \mathcal{F} be a cell as in the above paragraph of dimension $|S| - 1$ and assume that $\beta_{\gamma f \rho_{(S-w)}}(\mathcal{F})$ attains the minimal value for all such \mathcal{F} . Then we can retract \mathcal{F} into $\partial \mathcal{F}$ along the direction of the geodesic $\gamma f \rho_{(S-w)}$. Repeat this process until $\mathcal{J}_{y,e,\mathbf{R}}$ is retracted onto a complex of dimension $|S| - 2$. \square

We let $K_{y,e} \subseteq \mathcal{P}$ be the set of all $\mathbf{R} \in \mathcal{P}$ such that $\mathcal{J}_{y,e,\mathbf{R}} \neq \emptyset$. If $e, \epsilon \in E_y$, and if $\mathcal{J}_{y,e,\mathbf{R}}$ and $\mathcal{J}_{y,\epsilon,\mathbf{R}}$ are each nonempty, then $e = e(y, \mathbf{R}) = \epsilon$. Therefore, if e and ϵ are distinct, we have $K_{y,e} \cap K_{y,\epsilon} = \emptyset$ so that if we let $K_y = \cup_{e \in E_y} K_{y,e} \subseteq \mathcal{P}$, then

$$K_y = \coprod_{e \in E_y} K_{y,e}$$

Lemma 29. *Given a vertex $y \in X_w$ and $e \in E_y$,*

$$W_{S-w,(\max_{\epsilon \in E_y} \{s_{\mathbf{Q}}^\epsilon\})} = W_{S-w,(s_{\mathbf{Q}}^e)} \cup \left(\coprod_{\mathbf{R} \in K_{y,e}} \mathcal{J}_{y,e(y,\mathbf{R}),\mathbf{R}} \right)$$

Furthermore

$$W_{S-w,(s_{\mathbf{Q}}^e)} \cap \left(\coprod_{\mathbf{R} \in K_{y,e}} \mathcal{J}_{y,e(y,\mathbf{R}),\mathbf{R}} \right) = \coprod_{\mathbf{R} \in K_{y,e}} \mathcal{S}_{e(y,\mathbf{R}),\mathbf{R}}$$

Proof. By definition, for all $\mathbf{R} \in \mathcal{P}$, we have that $\mathcal{J}_{y,e,\mathbf{R}}$ and $W_{S-w,(s_{\mathbf{Q}}^e)}$ are contained in $W_{S-w,(\max_{\epsilon \in E_y} \{s_{\mathbf{Q}}^\epsilon\})}$.

If $\mathcal{F} \subseteq W_{S-w,(\max_{\epsilon \in E_y} \{s_{\mathbf{Q}}^\epsilon\})}$ is a cell, and if \mathcal{F} is not contained in $W_{S-w,(s_{\mathbf{Q}}^e)}$, then the maximum value of $\beta_{\gamma f \rho_{(S-w)}}(\mathcal{F})$ is greater than $s_{\mathbf{R}}^e$ for some $\mathbf{R} \in \mathcal{P}$ and $\gamma f \in \Lambda_{\mathbf{R}}$, so that $\mathcal{F} \subseteq \mathcal{J}_{y,e,\mathbf{R}}$ which is to say that

$$W_{S-w,(\max_{\epsilon \in E_y} \{s_{\mathbf{Q}}^\epsilon\})} \subseteq W_{S-w,(s_{\mathbf{Q}}^e)} \cup \bigcup_{\mathbf{R} \in \mathcal{P}} \mathcal{J}_{y,e,\mathbf{R}}$$

so we have equality. Furthermore, by the definition of $K_{y,e}$, and since $(s_{\mathbf{Q}}^e)_{\mathbf{Q} \in \mathcal{P}}$ is sufficiently large, we have

$$W_{S-w,(\max_{\epsilon \in E_y} \{s_{\mathbf{Q}}^\epsilon\})} = W_{S-w,(s_{\mathbf{Q}}^e)} \cup \coprod_{\mathbf{R} \in K_{y,e}} \mathcal{J}_{y,e,\mathbf{R}}$$

Now suppose that there is a cell \mathcal{F} contained in both $W_{S-w,(s_{\mathbf{Q}}^e)}$ and $\mathcal{J}_{y,e,\mathbf{R}}$ for some $\mathbf{R} \in K_{y,e}$. The latter inclusion implies that there is some $\mathcal{G} \subseteq X_{S-w}$ such that the maximum value of $\beta_{\gamma f \rho_{(S-w)}}(\mathcal{G})$ is greater than $s_{\mathbf{R}}^e$ for $\gamma f \in \Lambda_{\mathbf{R}}$. That is, $\mathcal{F} \subseteq \mathcal{S}_{e,\mathbf{R}}$.

To show the other inclusion, let $\mathcal{F} \subseteq W_{S-w,(s_{\mathbf{Q}}^e)}$ be such that there is a cell $\mathcal{G} \subseteq X_{S-w}$ containing \mathcal{F} with $\beta_{\gamma f \rho_{(S-w)}}(\mathcal{G}) \not\leq s_{\mathbf{R}}^e$ for some $\mathbf{R} \in K_{y,e}$ where $\gamma f \in \Lambda_{\mathbf{R}}$. Then $\mathcal{F} \subseteq \mathcal{J}_{y,e,\mathbf{R}}$. \square

We also have the following lemma whose proof is similar.

Lemma 30. *Given a vertex $y \in X_w$,*

$$W_{S-w,(\max_{\epsilon \in E_y} \{s_{\mathbf{Q}}^\epsilon\})} = W_{S-w,(s_{\mathbf{Q}}^{e(y,\mathbf{Q})})} \cup \left(\coprod_{\mathbf{R} \in K_y} \mathcal{J}_{y,e(y,\mathbf{R}),\mathbf{R}} \right)$$

Furthermore

$$W_{S-w,(s_{\mathbf{Q}}^{e(y,\mathbf{Q})})} \cap \left(\coprod_{\mathbf{R} \in K_y} \mathcal{J}_{y,e(y,\mathbf{R}),\mathbf{R}} \right) = \coprod_{\mathbf{R} \in K_y} \mathcal{S}_{e(y,\mathbf{R}),\mathbf{R}}$$

The Mayer-Vietoris sequence for the pair in Lemma 29 yields the coboundary homomorphism

$$\delta_{y,e} : \oplus_{\mathbf{R} \in K_{y,e}} H_c^{|S|-2}(\mathcal{S}_{e(y,\mathbf{R}),\mathbf{R}}) \rightarrow H_c^{|S|-1}(W_{S-w,(\max_{\epsilon \in E_y} \{s_{\mathbf{Q}}^\epsilon\})})$$

Similarly, Lemma 30 yields

$$\delta_y : \oplus_{\mathbf{R} \in K_y} H_c^{|S|-2}(\mathcal{S}_{e(y,\mathbf{R}),\mathbf{R}}) \rightarrow H_c^{|S|-1}(W_{S-w,(\max_{\epsilon \in E_y} \{s_{\mathbf{Q}}^\epsilon\})})$$

so that $\delta_y = \oplus_{e \in E_y} \delta_{y,e}$.

Lemma 31. *Suppose that $H_c^{|S|-2}(W_{S-w, (r_Q)}) = 0$ for any sufficiently large sequence $(r_Q)_{Q \in \mathcal{P}}$. If $\sum_{\mathbf{R} \in K_y} v_{\mathbf{R}} \in \oplus_{\mathbf{R} \in K_y} H_c^{|S|-2}(\mathcal{S}_{e(y, \mathbf{R}), \mathbf{R}})$ and $\delta_y(\sum v_{\mathbf{R}}) = 0$, then $\delta_y(v_{\mathbf{R}}) = 0$ for all $\mathbf{R} \in K_y$.*

Proof. Since $H_c^{|S|-2}(W_{S-w, (\max_{\epsilon \in E_y} s_{\mathbf{Q}}^\epsilon)})$ and $H_c^{|S|-2}(W_{S-w, (s_{\mathbf{Q}}^{e(y, \mathbf{Q})})})$ are trivial by assumption, we have the following portion of the Mayer-Vietoris sequence for the pair from Lemma 30.

$$0 \rightarrow \oplus H_c^{|S|-2}(\mathcal{J}_{y, e(y, \mathbf{R}), \mathbf{R}}) \rightarrow \oplus H_c^{|S|-2}(\mathcal{S}_{e(y, \mathbf{R}), \mathbf{R}}) \rightarrow H_c^{|S|-1}(W_{S-w, (\max_{\epsilon \in E_y} \{s_{\mathbf{Q}}^\epsilon\})})$$

where δ_y is the rightmost map on the line above. Thus, if $\delta_y(\sum v_{\mathbf{R}}) = 0$, then $\sum v_{\mathbf{R}} \in \oplus H_c^{|S|-2}(\mathcal{J}_{y, e(y, \mathbf{R}), \mathbf{R}})$, and in particular, for each \mathbf{R} we have $v_{\mathbf{R}} \in H_c^{|S|-2}(\mathcal{J}_{y, e(y, \mathbf{R}), \mathbf{R}})$ so that $\delta_y(v_{\mathbf{R}}) = 0$ for each \mathbf{R} . \square

Lemma 32. *Suppose that $H_c^{|S|-2}(W_{S-w, (r_Q)}) = 0$ for any sufficiently large sequence $(r_Q)_{Q \in \mathcal{P}}$. Let $y \in X_w$ be a vertex, and suppose $x \in H_c^{|S|-1}(F_y)$ is nonzero. Then there is at most one $e \in E_y$ such that $\rho_{y, e}(x) = 0$.*

Proof. Suppose that $\rho_{y, e}(x) = 0$, and let $\epsilon \in E_y$. We will show that $\rho_{y, \epsilon}(x) = 0$ implies $e = \epsilon$, thus proving the lemma. Applying the Mayer-Vietoris sequence to the sets from Lemma 29, and using Lemma 28, we have

$$\oplus_{\mathbf{R} \in K_{y, e}} H_c^{|S|-2}(\mathcal{S}_{e, \mathbf{R}}) \rightarrow H_c^{|S|-1}(W_{S-w, (\max_{\epsilon \in E_y} \{s_{\mathbf{Q}}^\epsilon\})}) \rightarrow H_c^{|S|-1}(W_{S-w, (s_{\mathbf{Q}}^e)})$$

where the map on the left is $\delta_{y, e}$ and the map on the right is $\rho_{y, e}$. Therefore, $\rho_{y, e}(x) = 0$ implies $x = \delta_{y, e}(\sum v_{\mathbf{R}}) = \delta_y(\sum v_{\mathbf{R}})$ for some $\sum v_{\mathbf{R}}$.

Now if $\rho_{y, \epsilon}(x) = 0$, then similarly, $x = \delta_y(\sum w_{\mathbf{R}})$ for some $\sum w_{\mathbf{R}}$. Therefore, $\delta_y(\sum (v_{\mathbf{R}} - w_{\mathbf{R}})) = x - x = 0$ which implies that $\delta_y(v_{\mathbf{R}} - w_{\mathbf{R}}) = 0$ for each \mathbf{R} by the previous lemma.

Now fix some \mathbf{R} with $\delta_{y, e}(v_{\mathbf{R}}) \neq 0$. Then

$$\delta_{y, \epsilon}(w_{\mathbf{R}}) = \delta_y(w_{\mathbf{R}}) = \delta_y(v_{\mathbf{R}}) = \delta_{y, e}(v_{\mathbf{R}}) \neq 0$$

from which we deduce that \mathbf{R} is contained in $K_{y, e}$ and $K_{y, \epsilon}$. Thus, $e = \epsilon$. \square

We are now ready to prove the main result of this section.

Proposition 33. *If $(r_Q)_{Q \in \mathcal{P}}$ is sufficiently large, then $H_c^k(W_{S, (r_Q)}) = 0$ if $k \neq |S|$ and thus $H_c^k(X_{S, (r_Q)}) = 0$ if $k \neq |S|$.*

Proof. If $|S| = 1$, then we have proved this lemma in Lemma 22, so we assume the lemma is true for $S - w$ and prove it is true for S .

By Lemma 27, and since for any vertex $y \in X_w$ we have $F_y = \cup_{e \in E_y} F_e$, we see that the collection $\{F_y\}$ taken over all vertices $y \in X_w$ is a cover for $W_{S,(r_Q)}$. Note also that if y and z are the endpoints of an edge $e \subseteq X_w$, then $F_e = F_y \cap F_z$. Thus, the nerve of $\{F_y\}$ can be identified with X_w , and there's an associated spectral with $E_2^{pq} = H_c^p(X_w, \{H_c^q(F_*)\})$ that converges to $H_c^{p+q}(W_{S,(r_Q)})$. (See, e.g. [7] VII.4 for the analogous homology sequence. The derivation of the sequence we use here is a straightforward adaptation of that one.)

Since $F_e = e \times W_{S-w, (s_Q^e)}$, our induction hypothesis implies that $H_c^q(F_e) = 0$ for $q \neq |S| - 1$ and, together with X_w being 1-dimensional, that implies $H_c^p(X_w, \{H_c^q(F_*)\}) = 0$ if $q \neq |S| - 1$ or if $p \geq 2$. Thus, we will have $H_c^k(W_{S,(r_Q)}) = 0$ for $k \neq |S|$ if $H_c^0(X_w, \{H_c^{|S|-1}(F_*)\}) = 0$, so we will verify that the kernel of the map

$$d : \oplus_{y \in X_w^{(0)}} H_c^{|S|-1}(F_y) \rightarrow \oplus_{e \in X_w^{(1)}} H_c^{|S|-1}(F_e)$$

is trivial.

To do this, suppose $\sum g_y \in \oplus H_c^{|S|-1}(F_y)$ is nonzero. Choose some vertex $y \in X_w$ with $g_y \neq 0$ and such that y is contained in an edge e and in the component of $X_w - e^\circ$ all of whose vertices $y' \neq y$ have $g_{y'} = 0$.

By Lemma 32, there is an edge $\epsilon \in E_y - e$ such that $\rho_{y,\epsilon}(g_y) \neq 0$. Therefore, the $H_c^{|S|-1}(F_\epsilon)$ component of $d(\sum g_y)$ is nonzero, and thus $d(\sum g_y) \neq 0$.

We have seen that $H_c^p(X_w, \{H_c^q(F_*)\}) = 0$ if $(p, q) \neq (1, |S| - 1)$. \square

If the sequence $(r_Q)_{Q \in \mathcal{P}}$ from Proposition 33 is constant, then we have the immediate

Corollary 34. *If $r \gg 0$ and $k \neq |S|$, then $H_c^k(X_{S,r}) = 0$.*

The proof of Proposition 33 given above applies to horosphere complements in products of trees that are more general than those arising from arithmetic groups. In particular, suppose $d \in \mathbb{N}$ and that T_i , for $1 \leq i \leq d$, is a locally finite tree with no vertices of valence 1. Choose geodesics $\Sigma_i : \mathbb{R} \rightarrow T_i$ parameaterized such that the integer values of Σ_i are exactly the vertices in T_i in the image of Σ_i . For any collection of positive numbers λ_i , let $\beta : \prod_{i=1}^d T_i \rightarrow \mathbb{R}$ be the Busemann function for $\rho : \mathbb{R} \rightarrow \prod_{i=1}^d T_i$ given by $\rho(t) = (\Sigma_i(\lambda_i t))_{i=1}^d$. Let $Z = \beta^{-1}((-\infty, r])$ for any given $r \in \mathbb{R}$.

Corollary 35. *If $k \neq d$, then $H_c^k(Z) = 0$.*

Proof. Apply Proposition 33 to a sequence $(r_Q)_{Q \in \mathcal{P}}$ that has exactly one finite value, and the result is the statement of this corollary. The

only exception is that Proposition 33 applies to trees whose valences are dictated by an arithmetic group, and it applies to Busemann functions for rays whose slopes (the λ_i) are determined by an arithmetic group. But neither of these explicit data are used in the proof of Proposition 33. \square

6. TOPOLOGY OF HOROSPHERES

Let $X = \prod_{i=1}^d T_i$ where each T_i is a locally finite tree with no vertices of valence 1. Suppose each edge length in T_i equals 1. For each tree T_i , choose a geodesic $\Sigma_i \subseteq T_i$ and label its vertices $x_{i,n}$ for $n \in \mathbb{Z}$. This induces a height function h_i on the vertices of T_i where $h_i(x_{i,0}) = 0$ and $h_i(v) = n - d(v, x_{i,n})$ if the closest vertex of Σ_i is $x_{i,n}$. Extend each h_i linearly over edges to produce a height function h_i defined on all of T_i . For $1 \leq i \leq d$, we choose $\lambda_i > 0$ and we define a Busemann function $\beta : X \rightarrow \mathbb{R}$ by $\beta(x_1, \dots, x_d) = \sum_i \lambda_i h_i(x_i)$.

Say that a vertex $v \in T_i$ is *below* a vertex $w \in T_i$ if there is a path γ from v to w such that $h_i \circ \gamma$ is strictly increasing. In this case we say w is *above* v . Note that for any $x_i \in T_i$ and $t > 0$ there is a unique point $y_i \in T_i$ above x_i such that $h_i(y_i) = h_i(x_i) + t$. Using this notation, the assignment $x_i \mapsto y_i$ defines a flow $\phi_{i,t} : T_i \rightarrow T_i$. These then define a flow on X by

$$\phi_t(x_1, \dots, x_d) = \left(\phi_{1,t/(\lambda_1 \sqrt{d})}(x_1), \dots, \phi_{d,t/(\lambda_d \sqrt{d})}(x_d) \right).$$

Note that $\beta(\phi_t(x)) = \beta(x) + t$.

For $r \in \mathbb{R}$, we define

$$\begin{aligned} Y_r &= \beta^{-1}(r) \\ X_r &= \beta^{-1}(-\infty, r], \text{ and} \\ B_r &= \beta^{-1}[r, \infty). \end{aligned}$$

The space X naturally has the structure of a cube complex. Subdivide this structure to give X the structure of a cell complex such that Y_r and X_r are subcomplexes. In particular, for each $(d-1)$ -cell e of Y_r there is a unique d -cell \hat{e} of X lying above e such that $\hat{e} \cap \hat{e} \neq \emptyset$.

In this section, all cohomology groups will be understood to have coefficients in some ring R .

6.1. Horoball cohomology.

Lemma 36. *For all r and k , $H_c^k(B_r) = 0$.*

Proof. For any number $m > n$ let

$$C(m) = \{ x = (x_1, \dots, x_d) \in X \mid \beta(x) \geq r \text{ and } x_i \text{ lies below } x_{i,m} \}$$

and let

$$\partial^\dagger C(m) = \{x \in C(m) \mid h_i(x) = m \text{ for some } i\}$$

Note that $C(m)$ deformation retracts onto $\partial^\dagger C(m)$. Thus we have that $H^k(C(m), \partial^\dagger C(m)) = 0$.

Note that the sets $C(m)$ form an exhaustion of B_r by compact sets. Let $c(m)$ be the closed subset of $C(m)$ consisting of points whose distance from $\partial^\dagger C(m)$ is at least ε , for some small $\varepsilon > 0$. The compact sets $c(m)$ also form an exhaustion of B_r , so it suffices to show $H^k(B_r, B_r - c(m)) = 0$ for all m . By excision we have

$$H^k(B_r, B_r - c(m)) \cong H^k(C(m), C(m) - c(m)).$$

Because $C(m) - c(m)$ deformation retracts onto $\partial^\dagger C(m)$, we have

$$H^k(C(m), C(m) - c(m)) \cong H^k(C(m), \partial^\dagger C(m)) = 0.$$

□

6.2. Horosphere cohomology. For $n \in \mathbb{N}$, the collection $\{H_c^{d-1}(Y_n)\}_{n \in \mathbb{N}}$ forms a directed system under the maps $(\phi_1)^* : H_c^{d-1}(Y_{n+1}) \rightarrow H_c^{d-1}(Y_n)$, where ϕ_1 is the time 1 flow on X . The goal of this section is to show $\varprojlim_n H_c^{d-1}(Y_n)$ is trivial and $\varprojlim_n^1 H_c^{d-1}(Y_n)$ is torsionfree.

There is a Mayer-Vietoris exact sequence

$$\cdots \rightarrow H_c^{d-1}(X_n) \oplus H_c^{d-1}(B_n) \rightarrow H_c^{d-1}(Y_n) \rightarrow H_c^d(X) \rightarrow \cdots$$

We know $H_c^{d-1}(X_n) = 0$ by Corollary 35 and $H_n^{d-1}(B_n) = 0$ by Lemma 36. Therefore the connecting map $H_c^{d-1}(Y_n) \rightarrow H_c^d(X)$ is injective. In this way we consider each module $H_c^{d-1}(Y_n)$ as a submodule of $H_c^d(X)$. Note that $(\phi_1)^* : H_c^d(X) \rightarrow H_c^d(X)$ is the identity map. It therefore follows from naturality of the Mayer-Vietoris sequence that $(\phi_1)^* : H_c^{d-1}(Y_{n+1}) \rightarrow H_c^{d-1}(Y_n)$ is the inclusion map of subgroups of $H_c^d(X)$.

We will prove $\varprojlim_n H_c^{d-1}(Y_n) = 0$, for which we set up notation. For any vertex $v \in T_i$ let $g_i(v)$ be the unique vertex above v such that $h_i(g_i(v)) = h_i(v) + 1$. Under the identification of $X^{(0)}$ with $\prod_i T_i^{(0)}$, let $g : X^{(0)} \rightarrow X^{(0)}$ be the function defined to be g_i in the i^{th} coordinate, so that $g(w)(i) = g_i(w(i))$ for any $w \in \prod_i T_i^{(0)}$.

Given i and n , let $C_{i,n} \subseteq T_i$ be the subtree of T_i spanned by the set of vertices

$$\{v \in T_i \mid v \text{ is below } x_{i,n} \text{ and } h_i(v) \geq -n\}.$$

Let $K_n = \prod_{i=1}^d C_{i,n}$. The collection $\{K_n\}_{n=0}^\infty$ forms an exhaustion of X by compact sets, and so $H_c^k(X) = \varinjlim_n H^k(X, X \setminus K_n)$.

We compute each relative cohomology group $H^k(X, X \setminus K_n)$ as the cohomology of the quotient space $X/(X \setminus K_n)$. Let $EC_{i,n}$ be the set of vertices $v \in C_{i,n}$ such that $h_i(v) = -n$. Each set $C_{i,n}$ is homotopy equivalent relative $EC_{i,n} \cup \{x_{i,n}\}$ to a star with $\#(EC_{i,n})$ leaves, and so if ∂K_n is comprised of points in K_n whose i^{th} coordinate for some i is contained in $EC_{i,n} \cup \{x_{i,n}\}$, then K_n is homotopy equivalent relative ∂K_n to a cube complex with a top dimensional cube for each vertex in $\prod_{i=1}^d EC_{i,n}$. It follows that there is a homotopy equivalence

$$X/(X \setminus K_n) \simeq \bigvee_{v \in \prod_{i=1}^d EC_{i,n}} S^d.$$

To simplify notation, let Λ_n be the set of vertices in $\prod_{i=1}^d EC_{i,n}$. With this notation, it follows that there is an isomorphism

$$H^k(X, X \setminus K_n) \cong \begin{cases} R^{\Lambda_n} & \text{if } k = d \\ 0 & \text{else.} \end{cases}$$

Under this identification, the map $f_n : R^{\Lambda_n} \rightarrow R^{\Lambda_{n+1}}$ induced by the map of pairs $(X, X \setminus K_{n+1}) \rightarrow (X, X \setminus K_n)$ is described as follows: Given a function $\alpha : \Lambda_n \rightarrow R$, define $f_n(\alpha)(w) = \alpha(g(w))$ if $g(w) \in \Lambda_n$ and $f_n(\alpha)(w) = 0$ otherwise.

Given a vertex $v \in \Lambda_n$, let $\overline{x_{i,n}, v(i)}$ denote the geodesic segment between $x_{i,n}$ and $v(i)$, and let F_v be the cube $\prod_{i=1}^d \overline{x_{i,n}, v(i)}$. Note there is an equality of spaces $K_n = \bigcup_{v \in \Lambda_n} F_v$. Given a compactly supported cellular cochain $\phi \in Z_c^d(X)$, the assignment $[\phi] \mapsto (v \mapsto \phi(F_v))_n$ gives the isomorphism $H_c^d(X) \cong \varinjlim_n R^{\Lambda_n}$.

Note that the above is a proof of the well-known

Proposition 37. $H_c^k(X) = 0$ if $k \neq d$ and $H_c^d(X)$ is a free R -module.

See Borel-Serre [6] for a more general theorem about the compactly supported cohomology of Euclidean buildings.

We now observe that $H_c^*(Y_r)$ is concentrated in dimension $d - 1$.

Proposition 38. $H_c^k(Y_r) = 0$ if $k \neq d - 1$ and $H_c^{d-1}(Y_r)$ is a free R -module.

Proof. There is a Mayer-Vietoris exact sequence

$$\cdots \rightarrow H_c^k(X_r) \oplus H_c^k(B_r) \rightarrow H_c^k(Y_r) \rightarrow H_c^{k+1}(X) \rightarrow \cdots$$

By Corollary 35 we know $H_c^k(X_r) = 0$ for $k \leq d - 1$. By Lemma 36 we know $H_c^k(B_r) = 0$ for all k . And by Proposition 37 we know $H_c^{k+1}(X) = 0$ for $k \leq d - 2$. The result follows. \square

Using the notation that we established prior to Proposition 37, we will prove

Lemma 39. $\varprojlim_n H_c^{d-1}(Y_n) = 0$.

Proof. Take any cohomology class $[\phi] \in H_c^d(X)$. Choose $n \in \mathbb{N}$ so that the support of ϕ is contained in K_n .

Consider any vertex $v \in \Lambda_n$, and choose m such that $m > \beta(K_{n+1})$. Suppose $[\phi] = [\delta\psi]$ for some $\psi \in H_c^{d-1}(Y_m)$, where δ is the chain map inducing the connecting homomorphism in the Mayer-Vietoris sequence. Choose $N > n + 1$ such that the support of ψ , and hence also the support of $\delta\psi$, is contained in K_N . Because $[\phi]$ and $[\delta\psi]$ are equal in $H_c^d(X)$ their images in R^{Λ_N} are equal.

Choose any $w \in \Lambda_N$ so that $g^{N-n}(w) = v$. Then $F_v \subseteq F_w$. Since the support of ϕ is contained in K_n and $F_v = F_w \cap K_n$ we have $\phi(F_v) = \phi(F_w)$.

For each $1 \leq i \leq d$ choose a vertex $e_i \in EC_{i,N}$ such that $g_i^{N-n-1}(e_i) \in EC_{i,n+1}$ but $g_i^{N-n} \notin EC_{i,n}$. Let $P(d)$ be the set of subsets of $\{1, \dots, d\}$. For each $\sigma \in P(d)$ define $w_\sigma \in \Lambda_n$ by

$$w_\sigma(i) = \begin{cases} w(i) & \text{if } i \notin \sigma \\ e_i & \text{if } i \in \sigma. \end{cases}$$

Note that $w_\emptyset = w$. If $\sigma \neq \emptyset$ then $F_{w_\sigma} \cap K_n = \emptyset$ and so $\phi(F_{w_\sigma}) = 0$. Because $[\phi]$ and $[\delta\psi]$ have the same image in R^{Λ_N} we see $\phi(F_{w_\sigma}) = \delta\psi(F_{w_\sigma})$ for all $\sigma \in P(d)$.

We claim the $(d-1)$ -chain $\sum_{\sigma \in P(d)} (-1)^{|\sigma|} (F_{w_\sigma} \cap Y_m)$ is the zero chain. Indeed, for $1 \leq i \leq d$, let u_i be an order 2 isometry of T_i with $u_i(w(i)) = e_i$ and $u_i(x_i) = x_i$ if $h_i(x_i) \geq n+1$. For $\sigma \in P(d)$ we let u_σ be the product of u_i with $i \in \sigma$. In particular, $u_\emptyset = 1$. Notice that $u_\sigma F_w = F_{w_\sigma}$, and that $u_\sigma Y_m = Y_m$.

We let $P(d)^* = P(d) - \{1, \dots, d\}$, and for each $\tau \in P(d)^*$, we define

$$R_\tau = \{(x_i) \in F_w \cap Y_m \mid h_i(x_i) \leq n+1 \text{ precisely when } i \in \tau\}$$

Thus, $\cup_{\tau \in P(d)^*} R_\tau = F_w \cap Y_m$, if $\tau_1 \neq \tau_2$ then R_{τ_1} and R_{τ_2} do not contain a common $(d-1)$ -cell, and if $\sigma \in P(d)$ then $u_\sigma R_\tau = u_{\sigma \cap \tau} R_\tau$.

Recall that by the binomial theorem, if $k \in \mathbb{N}$, then $\sum_{\mu \in P(k)} (-1)^{|\mu|} = 0$. Thus we have

$$\begin{aligned}
\sum_{\sigma \in P(d)} (-1)^{|\sigma|} (F_{w_\sigma} \cap Y_m) &= \sum_{\sigma \in P(d)} (-1)^{|\sigma|} (u_\sigma F_w \cap Y_m) \\
&= \sum_{\sigma \in P(d)} (-1)^{|\sigma|} u_\sigma (F_w \cap Y_m) \\
&= \sum_{\sigma \in P(d)} (-1)^{|\sigma|} u_\sigma \left(\sum_{\tau \in P(d)^*} R_\tau \right) \\
&= \sum_{\tau \in P(d)^*} \sum_{\sigma \in P(d)} (-1)^{|\sigma|} u_\sigma R_\tau \\
&= \sum_{\tau \in P(d)^*} \sum_{\sigma \in P(d)} (-1)^{|\sigma|} u_{\sigma \cap \tau} R_\tau \\
&= \sum_{\tau \in P(d)^*} \sum_{\rho \in P(|\tau|)} \sum_{\mu \in P(d-|\tau|)} (-1)^{|\rho|+|\mu|} u_\rho R_\tau \\
&= \sum_{\tau \in P(d)^*} \sum_{\rho \in P(|\tau|)} (-1)^{|\rho|} \sum_{\mu \in P(d-|\tau|)} (-1)^{|\mu|} u_\rho R_\tau \\
&= \sum_{\tau \in P(d)^*} \sum_{\rho \in P(|\tau|)} (-1)^{|\rho|} 0 u_\rho R_\tau \\
&= 0
\end{aligned}$$

This establishes our claim that $\sum_{\sigma \in P(d)} (-1)^{|\sigma|} (F_{w_\sigma} \cap Y_m)$ is the zero chain.

Using the definition of the connecting map δ we therefore have

$$\begin{aligned}
0 &= \psi \left(\sum_{\sigma \in P(d)} (-1)^{|\sigma|} F_{w_\sigma} \cap Y_\sigma \right) \\
&= \sum_{\sigma \in P(d)} (-1)^{|\sigma|} \delta \psi(F_{w_\sigma}) \\
&= \sum_{\sigma \in P(d)} (-1)^{|\sigma|} \phi(F_{w_\sigma}) \\
&= \phi(F_w) \\
&= \phi(F_v).
\end{aligned}$$

This shows that if $[\phi] \in H_c^d(X)$, if $[\phi]$ is contained in $H_c^{d-1}(Y_m) \leq H_c^d(X)$ for some sufficiently large value of m , then $[\phi] = 0$. That is, $\varprojlim_n H_c^{d-1}(Y_n) = \cap_m H_c^{d-1}(Y_m) = 0$. \square

Our next goal is to prove that $\varprojlim^1 H_c^{d-1}(Y_m)$ is torsion-free.

Lemma 40. *Suppose $[\psi] \in H_c^{d-1}(Y_m)$ and suppose there are $[\phi] \in H_c^d(X)$ and $r \in R$ such that $r[\phi] = [\delta\psi]$. Then there is some $[\tilde{\phi}] \in H_c^{d-1}(Y_m)$ such that $r[\tilde{\phi}] = [\psi]$.*

Proof. Let \hat{Y}_m be the subcomplex of X containing only the (subdivided) d -cells $e \subseteq X$ such that $\beta(e) \geq m$ and $e \cap Y_m \neq \emptyset$. Let $i : \hat{Y}_m \rightarrow X$ be the inclusion map. The Mayer-Vietoris connecting map induces a homomorphism $\hat{\delta} : H_c^{d-1}(Y_m) \rightarrow H_c^d(\hat{Y}_m)$ such that the following diagram commutes:

$$\begin{array}{ccc} & H_c^{d-1}(Y_m) & \\ \delta \swarrow & & \searrow \hat{\delta} \\ H_c^d(X) & \xrightarrow{i^*} & H_c^d(\hat{Y}_m) \end{array}$$

Given any $(d-1)$ -cell $c \subseteq Y_m$, let $\hat{c} \subseteq \hat{Y}_m$ be the unique d -cell in \hat{Y}_m with $c \subseteq \hat{c}$.

If $[\phi] \in H_c^d(\hat{Y}_m)$, then define $[\epsilon\phi] \in H_c^{d-1}(Y_m)$ by $\epsilon\phi(c) = \phi(\hat{c})$. Then $\epsilon : H_c^d(\hat{Y}_m) \rightarrow H_c^{d-1}(Y_m)$ is the inverse of $\hat{\delta}$, so there is an isomorphism $H_c^{d-1}(Y_m) \cong H_c^d(\hat{Y}_m)$. The lemma follows by setting $[\tilde{\phi}] = \epsilon i^*([\phi])$. \square

Lemma 41. $\varprojlim^1 H_c^{d-1}(Y_m)$ is torsionfree as an R -module.

Proof. Recall that $\varprojlim^1 H_c^{d-1}(Y_m)$ is the cokernel of the map

$$\Delta : \prod_m H_c^{d-1}(Y_m) \rightarrow \prod_m H_c^{d-1}(Y_m)$$

where $\Delta(([\psi_m])_m) = ([\psi_m] - [\psi_{m+1}])_m$.

Suppose $([\xi_m])_m \in \prod_m H_c^{d-1}(Y_m)$ and that there is some regular element $r \in R$ such that $(r[\xi_m])_m$ is in the image of Δ . Let $([\psi_m])_m \in \prod_m H_c^{d-1}(Y_m)$ be a sequence such that $r[\xi_m] = [\psi_m] - [\psi_{m+1}]$ for all m . Note that this implies that for any $M > m$ there is some $\zeta_{m,M} \in H_c^{d-1}(Y_m)$ so that $r[\zeta_{m,M}] = [\psi_m] - [\psi_M]$.

Fix any m and choose $n \in \mathbb{N}$ such that ψ_m is supported on K_n . Take any $v \in \Lambda_n$. Choose any $M > m$ such that $M > \beta(K_n)$. Choose $N \in \mathbb{N}$ such that ψ_M is supported on K_N .

Choose any $w \in \Lambda_N$ such that $g^{N-n}(w) = v$. Construct vertices $w_\sigma \in \Lambda_N$ for each $\sigma \in P(d)$ as in the proof of Lemma 39, so that $F_{w_0} = F_v$ and $\psi_m(F_{w_\sigma}) = 0$ if $\sigma \neq \emptyset$ and $\sum_{\sigma \in P(d)} (-1)^{|\sigma|} F_{w_\sigma} \cap Y_M$ is

the zero chain. Then

$$\psi_M \left(\sum_{\sigma \in P(d)} (-1)^{|\sigma|} F_{w_\sigma} \cap Y_M \right) = 0$$

so

$$\delta \psi_M \left(\sum_{\sigma \in P(d)} (-1)^{|\sigma|} F_{w_\sigma} \right) = 0.$$

It follows that r divides the quantity

$$\begin{aligned} & (\delta \psi_m - \delta \psi_M) \left(\sum_{\sigma \in P(d)} (-1)^{|\sigma|} F_{w_\sigma} \right) \\ &= \delta \psi_m \left(\sum_{\sigma \in P(d)} (-1)^{|\sigma|} F_{w_\sigma} \right) - \delta \psi_M \left(\sum_{\sigma \in P(d)} (-1)^{|\sigma|} F_{w_\sigma} \right) \\ &= \delta \psi_m(F_{w_\emptyset}) \\ &= \psi_m(F_v \cap Y_M). \end{aligned}$$

This holds for any vertex $v \in \Lambda_n$, so r divides the image of $[\psi_m]$ in R^{Λ_n} . Therefore r divides the image of $[\psi_m]$ in $H_c^d(X)$. By Lemma 40, for each m there is some $[\phi_m] \in H_c^{d-1}(Y_m)$ such that $r[\phi_m] = [\psi_m]$. It follows that $([\xi_m])_m = \Delta([\phi_m])$, which completes the proof. \square

7. EXAMPLES OF SEMIDUALITY GROUPS

Below we provide examples of semiduality groups. In order to verify the condition on the cohomological dimension, we recall the following standard result.

Lemma 42. *Suppose Λ is a group acting on an acyclic cell complex X with finite cell stabilizers. Suppose R is a commutative ring such that $|\Lambda_\sigma|$ is invertible in R for any cell stabilizer Λ_σ . Then $\text{cd}_R \Lambda \leq \dim(X)$.*

Proof. Suppose M is a $R\Lambda$ -module. For each j let Σ_j be a set representatives of Λ -orbits of j -cells of X . There is a spectral sequence (compare to the homology version appearing in [7, VII.7.7, p173])

$$E_1^{jq} = \prod_{\sigma \in \Sigma_j} H^q(\Lambda_\sigma; M) \implies H^{j+q}(\Lambda; M).$$

For any $q > 0$ the module $H^q(\Lambda_\sigma, M)$ is annihilated by $|\Lambda_\sigma|$. But M is an R -module and $|\Lambda_\sigma|$ is invertible in R for any cell σ , so the groups $H^q(\Lambda_\sigma; M)$ are trivial if $q > 0$. By definition, the groups E_1^{jq} are trivial for $j > \dim(X)$. It follows that $\text{cd}_R \Lambda \leq \dim(X)$. \square

7.1. Rank 1 arithmetic groups. In this section we prove Theorem 3. To that end, suppose \mathcal{O}_S is the ring of S -integers in a global function field K of characteristic p . Suppose \mathbf{G} is a noncommutative, absolutely almost simple algebraic K -group and Γ is a finite-index subgroup of $\mathbf{G}(\mathcal{O}_S)$ such that any torsion element of Γ is a p -element.

Proposition 43. *Γ is a $\mathbb{Z}[1/p]$ -semiduality group of dimension $k(\mathbf{G}, S)$.*

Proof. Γ acts on the product of trees X_S with finite p -group stabilizers. It follows from Lemma 42 that $\text{cd}_{\mathbb{Z}[1/p]} \Gamma \leq \dim(X_S) = k(\mathbf{G}, S)$. It is known that Γ is type $FP_{k(\mathbf{G}, S)-1}$ over $\mathbb{Z}[1/p]$ (see [17], [12], [10]). It remains to show that $H^*(\Gamma, \mathbb{Z}[1/p]\Gamma)$ is concentrated in dimension $k(\mathbf{G}, S)$, where it is flat as a $\mathbb{Z}[1/p]$ -module.

For sufficiently large $n \in \mathbb{N}$, Γ acts properly cocompactly on each $X_{S,n}$ by Lemma 19. As n tends to infinity, the spaces $X_{S,n}$ exhaust X_S . By Proposition 11 there is an isomorphism $H_{c\uparrow}^k(X_S) \cong H^k(\Gamma, \mathbb{Z}[1/p]\Gamma)$, where here and for the rest of the proof we take cohomology of spaces with $\mathbb{Z}[1/p]$ coefficients. We proved in Proposition 33 that $H_c^*(X_{S,n})$ is concentrated in dimension d . It follows from Proposition 13 that $H_{c\uparrow}^*(X_S)$ is concentrated in dimension d , where $H_{c\uparrow}^d(X_S) = \varprojlim_n H_c^d(X_{S,n})$.

It remains only to show that $\varprojlim_n H_c^d(X_{S,n})$ is $\mathbb{Z}[1/p]$ -torsion-free, since $\mathbb{Z}[1/p]$ is a principal ideal domain. The closure of the complement of $X_{S,n}$ is a disjoint union of horoballs $\bigcup_{\mathbf{Q} \in \mathcal{P}} B_{\mathbf{Q}, S, n}$. Up to proper homotopy, each set $B_{\mathbf{Q}, S, n}$ is a set of the form B_n as defined in §6, and $X_{S,n} \cap B_n = Y_n$, where $Y_n = Y_{\mathbf{Q}, S, n}$. There is a Mayer-Vietoris exact sequence

$$\begin{aligned} H_c^{d-1}(X_{S,n}) \oplus H_c^{d-1}\left(\bigcup_{\mathcal{P}} B_n\right) &\rightarrow H_c^{d-1}\left(\bigcup_{\mathcal{P}} Y_n\right) \rightarrow \\ &H_c^d(X_S) \rightarrow H_c^d(X_{S,n}) \oplus H_c^d\left(\bigcup_{\mathcal{P}} B_n\right) \rightarrow H_c^d\left(\bigcup_{\mathcal{P}} Y_n\right). \end{aligned}$$

Because the unions are disjoint, for each k there are isomorphisms

$$H_c^k\left(\bigcup_{\mathcal{P}} B_n\right) \cong \bigoplus_{\mathcal{P}} H_c^k(B_n) \quad \text{and} \quad H_c^k\left(\bigcup_{\mathcal{P}} Y_n\right) \cong \bigoplus_{\mathcal{P}} H_c^k(Y_n).$$

We know $H_c^k(B_n) = 0$ for all k by Lemma 36. We also know $H_c^{d-1}(X_{S,n}) = 0$ by Proposition 33. Clearly $H_c^d(Y_n) = 0$ since Y_n is $(d-1)$ -dimensional. Therefore we have a short exact sequence

$$0 \rightarrow \bigoplus_{\mathcal{P}} H_c^{d-1}(Y_n) \rightarrow H_c^d(X_S) \rightarrow H_c^d(X_{S,n}) \rightarrow 0.$$

These maps are compatible with the maps induced by inclusion $i_n : X_{S,n} \rightarrow X_{S,n+1}$, the time 1 flow $\phi_1 : Y_n \rightarrow Y_{n+1}$, and the identity

map $X_S \rightarrow X_S$. The above short exact sequence therefore gives rise to a short exact sequence of codirected systems of compactly support cohomology, from which there is an exact sequence

$$0 \rightarrow \varprojlim \bigoplus_{\mathcal{P}} H_c^{d-1}(Y_n) \rightarrow H_c^d(X_S) \rightarrow \varprojlim H_c^d(X_{S,n}) \rightarrow \varprojlim^1 \bigoplus_{\mathcal{P}} H_c^{d-1}(Y_n) \rightarrow 0.$$

The maps of the system $\{\bigoplus H_c^{d-1}(Y_n)\}$ preserve the direct sum structure. We know $\varprojlim H_c^{d-1}(Y_n)$ is trivial by Proposition 39 and $\varprojlim^1 H_c^{d-1}(Y_n)$ is torsionfree by Proposition 41. Since $H_c^d(X_S)$ is torsionfree by Proposition 37, it follows that $\varprojlim H_c^d(X_{S,n})$ is torsionfree by the short exact sequence

$$(3) \quad 0 \rightarrow H_c^d(X_S) \rightarrow \varprojlim H_c^d(X_{S,n}) \rightarrow \bigoplus_{\mathcal{P}} \varprojlim^1 H_c^{d-1}(Y_n) \rightarrow 0.$$

□

This proves Theorem 3, as every module in sequence (3) is a $\mathbb{Z}[1/p]\mathbf{G}(K)$ -module by Lemma 12.

7.2. Solvable groups. In this section we prove groups of the form $\mathbf{B}_2(\mathcal{O}_S)$ are semiduality groups. We then prove that some generalizations of certain groups of this form are also semiduality groups, namely lamplighter groups, Diestel-Leader groups, and countable direct sums of finite groups. All are straightforward applications of the following lemma.

Lemma 44. *Let X be a product of d trees with Busemann function $\beta : X \rightarrow \mathbb{R}$ as described in §6. Suppose a group Λ acts on X cellularly, with finite cell stabilizers, and cocompactly on subsets of the form $\beta^{-1}(I)$ for closed intervals I . Suppose R is a principal ideal domain such that $|\Lambda_\sigma|$ is invertible for every cell stabilizer Λ_σ . Then Λ is an R -semiduality group of dimension d .*

Proof. Define

$$\begin{aligned} Y_n &= \beta^{-1}(\{n\}) \\ X_n &= \beta^{-1}[0, n], \text{ and} \\ B_n &= \beta^{-1}[n, \infty). \end{aligned}$$

The space X is contractible, so by Lemma 42 we know $\text{cd}_R \Lambda \leq \dim(X) = d$. Since Λ acts cocompactly with finite stabilizers on a horosphere Y_n and $\tilde{H}_k(Y_n) = 0$ for $k < n - 1$ by [9, 3.1], Brown's criterion implies that Λ is type FP_{d-1} (see for example [8, 1.1]).

The complexes X_n form an exhaustion of B_0 by closed, Γ -invariant sets such that $\Gamma \backslash X_n$ is compact. Therefore by the results of §3 there is

an isomorphism $H^*(\Lambda; R\Lambda) \cong H_{c\uparrow}^k(B_0)$ and for each k there is an exact sequence

$$(4) \quad 0 \rightarrow \varprojlim^1 H_c^{k-1}(X_n) \rightarrow H_{c\uparrow}^k(B_0) \rightarrow \varprojlim H_c^k(X_n) \rightarrow 0.$$

Note that the flow ϕ_t provides a proper deformation retraction of X_n to Y_n so $H_c^*(X_n) \cong H_c^*(Y_n)$. We know $H_c^*(Y_n)$ is concentrated in dimension $d-1$ by Proposition 38. Now Lemma 39 says $\varprojlim H_c^{d-1}(Y_n) = 0$ so $H_{c\uparrow}^*(B_0)$ is concentrated in dimension d . In that dimension there is an isomorphism $H_{c\uparrow}^d(B_0) \cong \varprojlim^1 H_c^{d-1}(Y_n)$, which is torsionfree by Lemma 41 and hence flat as an \hat{R} -module. \square

Suppose \mathcal{O}_S is the ring of S -integers in a global function field K of characteristic p . Let \mathbf{B}_2 be the group of upper triangular matrices of determinant 1.

Theorem 45. *Suppose Γ is a finite index subgroup of $\mathbf{B}_2(\mathcal{O}_S)$ such that the order of every finite order element is a power of p . Then Γ is a $\mathbb{Z}[1/p]$ -semiduality group of dimension $|S|$.*

Proof. In the notation of §4, we may choose $\mathbf{P} = \mathbf{B}_2$. Then applying Lemma 17 with $\gamma = 1$ and $f = 1$, we see that \mathbf{P} acts on the horoball $B_{\mathbf{P},S,n}$ for all sufficiently large $n \in \mathbb{N}$. In fact $\mathbf{P}(\mathcal{O}_S)$ is the entire stabilizer of $B_{\mathbf{P},S,n}$ in $\mathbf{SL}_2(\mathcal{O}_S)$ since if $\gamma \in \mathbf{SL}_2(\mathcal{O}_S)$ and $\gamma B_n = B_n$ then $B_{\mathbf{P},S,n} = B_{\gamma\mathbf{P}\gamma^{-1},S,n}$, which by Proposition 18(ii) means $\mathbf{P} = \gamma\mathbf{P}\gamma^{-1}$ and so $\gamma \in \mathbf{P}$. It follows that the action of \mathbf{P} on $Y_{\mathbf{P},S,n}$ is proper and cocompact since the action of $\mathbf{SL}_2(\mathcal{O}_S)$ is proper and cocompact on X_S . In particular, cell stabilizers are finite.

Γ acts on the product of trees X_S . Let β be the Busemann function associated to the end \mathbf{P} . By the previous paragraph Γ acts cocompactly on $\beta^{-1}(I)$ for any compact interval $I \subset \mathbb{R}$ because it has finite index in $\mathbf{P}(\mathcal{O}_S)$. Since Γ has only p -torsion and its action is proper, Lemma 44 applies. \square

Suppose F is a finite group. The *lamplighter group with base group F* is $\Gamma_F = F \wr \mathbb{Z} = (\oplus_{i \in \mathbb{Z}} F) \rtimes \mathbb{Z}$, where \mathbb{Z} acts by shifting the indices of a sequence (f_i) .

Theorem 46. *The lamplighter group with base group F is a $\mathbb{Z}[1/|F|]$ -semiduality group of dimension 2.*

Proof. Let T_1 and T_2 be copies of a $(|F|+1)$ -regular tree. The lamplighter group Γ_F acts on $T_1 \times T_2$ in a natural way; for description of the action see [19, §4]. This action preserves a Busemann function β and is cocompact on any set of the form $\beta^{-1}(I)$ for closed intervals $I \subseteq \mathbb{R}$.

Stabilizers of cells are finite sums of copies of F . Therefore Lemma 44 applies. \square

There are “higher rank” generalizations of lamplighter groups known as *Diestel-Leader groups* $\Gamma_d(q)$ which act on a product of d regular trees of valence $q + 1$. These are constructed in [1] for any values of d and q such that $d \leq p + 1$ for any prime p dividing q ; a lamplighter group with base group F is an example of $\Gamma_2(|F|)$. The proof of Theorem 46 easily generalizes to prove:

Theorem 47. *A Diestel-Leader group $\Gamma_d(q)$ is a $\mathbb{Z}[1/q]$ -semiduality group of dimension d .*

As a final remark, consider a countable collection of finite groups $\{F_i\}_{i \in \mathbb{N}}$ and let $\Lambda = \bigoplus_{i \in \mathbb{N}} F_i$. (This is not necessarily solvable.) Then Λ is an R -semiduality group of dimension 1 for any principal ideal domain R in which $|F_i|$ is invertible for every i . (So, for example, any countable sum of finite groups is a \mathbb{Q} -semiduality group.) To see this, let $\Lambda_n = \bigoplus_{i=0}^n F_i$. Form a graph of groups with underlying graph a simplicial ray whose n th vertex and proceeding edge are labeled by Λ_n , with inclusion maps from edge groups to incident vertex groups. Then Λ is the fundamental group of this graph of groups. It acts on the Bass-Serre tree preserving a height function inherited from the base ray, and is cocompact on preimages of closed intervals. Cell stabilizers are isomorphic to some Λ_n , so Lemma 44 produces the desired result.

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